Topos Theory - Solutions to Exercise Sheet 3

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Before solving the exercises, we record three useful facts concerning initial objects in cartesian closed categories.

Proposition 1. *In a cartesian closed category $\mathcal{C}$ with an initial object 0, the product $0 \times X$ is initial for any object X of $\mathcal{C}$.*

Proof. For arbitrary objects X and Y of $\mathcal{C}$, we have $\mathrm{Hom}(0 \times X, Y) \cong \mathrm{Hom}\big(0, Y^X\big)$, since $\mathcal{C}$ is cartesian closed. But the latter hom-set is a singleton, because 0 is initial. Hence, $\mathrm{Hom}(0 \times X, Y)$ is a singleton too for every object Y, which says exactly that $0 \times X$ is initial. □

Alternative proof. In a cartesian closed category the functor $X \times (-)$ is a left adjoint, hence preserves all colimits and the initial object in particular. □

Proposition 2. *In a cartesian closed category $\mathcal{C}$ the initial object (if it exists) is* strict*: any map with 0 as its codomain is an isomorphism. Moreover, there is at most one map from any object to the initial object.*

Proof. Suppose that we have $f\colon X \to 0$. It follows that f is an isomorphism, if we can prove that X is initial. By Proposition 1, it suffices to prove that X and $0 \times X$ are isomorphic. Writing $!_X$ for the unique map from 0 to X, we see that the composites

$$0 \times X \xrightarrow{\quad \pi_2 \quad} X \xrightarrow{(!_X, \mathrm{id}_X)} 0 \times X$$

$$X \xrightarrow{(!_X, \mathrm{id}_X)} 0 \times X \xrightarrow{\quad \pi_2 \quad} X$$

both equal the respective identity morphisms (the top one by initiality of $0 \times X$).

The second claim in the lemma follows from the fact that if we had two such maps with domain X, then both of them must be the inverse of the unique map from 0 to X. □

Corollary 3. *In a cartesian closed category with initial object 0, the unique morphism from 0 to any object X is a mono.*

Proof. By the second part of Proposition 2. □

Solutions to the exercises

1. The partial order on $\mathrm{Sub}(X)$ is defined as follows. A subobject represented by $m\colon A \rightarrowtail X$ is less than a subobject represented by $n\colon B \rightarrowtail X$ if there exists a dashed map (which is

necessarily a mono) making the diagram

$$\begin{tikzcd}
A \arrow[rr, dashed] \arrow[dr, tail, "m"'] & & B \arrow[dl, tail, "n"] \\
& X &
\end{tikzcd}$$

commute. The dashed map is unique (if it exists), because n is a mono.

Note that this order is transitive by virtue of composition and reflexive because of identity morphisms.

The order is antisymmetric, for if we had $f\colon A \to B$ and $g\colon B \to A$ such that $n \circ f = m$ and $m \circ g = n$, then $n \circ f \circ g = m \circ g = n \circ \mathrm{id}$, so that $f \circ g = \mathrm{id}$, as n is a mono. Similarly, $g \circ f = \mathrm{id}$, and hence, A and B must represent the same subobject in this case.

The greatest element of $\mathrm{Sub}(X)$ is given by (the subobject represented by) the identity on X, which is a mono, of course. The least element of $\mathrm{Sub}(X)$ is given by (the subobject represented by) the unique map from 0 to X, which is a mono by Proposition 3.

Meets in $\mathrm{Sub}(X)$ are calculated as follows. Given two subobjects of X, represented by $A \rightarrowtail X$ and $B \rightarrowtail X$, respectively, their meet is the subobject represented by the pullback

$$\begin{tikzcd}
A \wedge B \arrow[r, tail] \arrow[d, tail] & A \arrow[d, tail] \\
B \arrow[r, tail] & X
\end{tikzcd}$$

This indeed gives the *greatest* lower bound exactly by the universal property of the pullback.

Joins are slightly more complicated. We first take the coproduct of A and B. The coproduct inclusions then yield a unique map from $A + B$ to X, which we factor through its image $M \rightarrowtail X$ (as explained in the lecture). The join of the subobjects is then given by the mono $M \rightarrowtail X$. The diagram below illustrates the situation.

$$\begin{tikzcd}
A + B \arrow[dr] & & A \arrow[ll] \arrow[dd, tail] \\
& M \arrow[dr, tail] & \\
B \arrow[uu] \arrow[rr, tail] & & X
\end{tikzcd}$$

The join is an upper bound, because of the coproduct inclusions and it is the *least* such one precisely because M is the smallest subobject of X through which $A + B \to X$ can factor.

Finally, for the construction of the Heyting implication, we make use of the observation that for any object Y of any topos $\mathcal{F}$, the posets $\mathrm{Sub}_{\mathcal{F}}(Y)$ and $\mathrm{Sub}_{\mathcal{F}/Y}(1)$ are isomorphic. Hence, for proving that $\mathrm{Sub}_{\mathcal{E}}(X)$ has a Heyting implication, it suffices to prove that the posets of *subterminals* (subobjects of 1) in an arbitrary topos has a Heyting implication.

So suppose that U and V are subterminals. Then $\mathrm{Hom}(Y, V)$ has at most one element for any object Y, exactly because V is subterminal and by the fact that there is a unique map into 1. Hence, $\mathrm{Hom}\big(Z, V^U\big) \cong \mathrm{Hom}(Z \times U, V)$ also has at most one element for any object Z, so V^U is subterminal too. Moreover, it is the Heyting implication of U and V because of the universal property of the exponential.

2. We first check that G^F with $G^F(C) \coloneqq \mathrm{Hom}_{\mathcal{S}et^{\mathcal{C}^{\mathrm{op}}}}(\mathbf{y}(C) \times F, G)$ is a presheaf. Given an arrow $f\colon D \to C$ in $\mathcal{C}$, we define the map $G^F(f)\colon G^F(C) \to G^F(D)$ as $\alpha \mapsto \alpha \circ (\mathbf{y}(f) \times \mathrm{id}_F)$, and we note that this action is clearly functorial.

   Next we define a natural transformation $\mathrm{eval}\colon G^F \times F \to G$. At an object C, we put $\mathrm{eval}_C(\gamma, x) \coloneqq \gamma_C(\mathrm{id}_C, x)$. We check that this is indeed a natural transformation. Suppose we have $\gamma \in G^F(C)$, $x \in F(C)$ and $f\colon D \to C$ in $\mathcal{C}$ and consider the diagram

$$\begin{tikzcd} G^F(C) \times F(C) \arrow[r, "\mathrm{eval}_C"] \arrow[d, "G^F(f)\times F(f)"'] & G(C) \arrow[d, "G(f)"] \\ G^F(D) \times F(D) \arrow[r, "\mathrm{eval}_D"] & G(D) \end{tikzcd}$$

   In the clockwise direction we get $G(f)(\gamma_C(\mathrm{id}_C, x))$, while the anti-clockwise direction yields $\gamma_D(f, F(f)(x))$. And these must be equal, because γ is a natural transformation from $\mathbf{y}(C) \times F$ to G, so the diagram

$$\begin{tikzcd} \mathrm{Hom}_{\mathcal{C}}(C, C) \times F(C) \arrow[r, "\gamma_C"] \arrow[d, "(-\circ f)\times F(f)"'] & G(C) \arrow[d, "G(f)"] \\ \mathrm{Hom}_{\mathcal{C}}(D, C) \times F(D) \arrow[r, "\gamma_D"] & G(D) \end{tikzcd}$$

   commutes.

   Given a natural transformation $\alpha\colon H \times F \to G$, we construct its transpose $\hat{\alpha}\colon H \to G^F$ as follows. At an object C, we define $\hat{\alpha}_C\colon H(C) \to G^F(C)$ by sending $y \in H(C)$ to the natural transformation given at an object D by:

$$\begin{aligned} (\hat{\alpha}_C^y)_D\colon \mathrm{Hom}_{\mathcal{C}}(D, C) \times F(D) &\to G(D) \\ (f \ , \ x) \quad &\mapsto \alpha_D(H(f)(y), x). \end{aligned}$$

   Then one can verify that $\hat{\alpha}$ and $\hat{\alpha}_C^y$ are indeed natural transformations using the naturality of α.

   We check that $\mathrm{eval} \circ (\hat{\alpha} \times \mathrm{id}_F) = \alpha$. Indeed, for an object C and elements $y \in H(C)$ and $x \in F(C)$, we calculate that

$$\big(\mathrm{eval}_C \circ \big(\hat{\alpha}_C \times \mathrm{id}_{F(C)}\big)\big)(y, x) = (\hat{\alpha}_C^y)_C(\mathrm{id}_C, x) = \alpha_C(H(\mathrm{id}_C)(y), x) = \alpha_C(y, x)),$$

   as desired.

   Finally, we show that $\hat{\alpha}$ is the unique such natural transformation. For suppose $\beta\colon H \to G^F$ is a natural transformation such that $\mathrm{eval} \circ (\beta \circ \mathrm{id}_F) = \alpha$. Then $\beta_C(y)(\mathrm{id}_C, x) = \alpha(y, x)$ for every object C, $y \in H(C)$ and $x \in F(C)$. By naturality of β, the diagram

$$\begin{tikzcd} H(C) \arrow[r, "\beta_C"] \arrow[d, "H(f)"'] & G^F(C) \arrow[d, "G^F(f)"] \\ H(D) \arrow[r, "\beta_D"] & G^F(D) \end{tikzcd}$$

commutes for every arrow $f\colon D \to C$ in $\mathcal{C}$. Hence, $G^F(f)(\beta_C(y)) = \beta_D(H(f)(y))$ for every element $y \in H(C)$. Putting the above together, we see that for every $f\colon D \to C$ and $x \in F(D)$ it holds that

$$\begin{aligned}
(\beta_C(y))_D(f,x) &= \left(G^F(f)(\beta_C(y))\right)_D(\mathrm{id}_D, x) && \text{(by definition of } G^F(f)\text{)}\\
&= \beta_D(H(f)(y))(\mathrm{id}_D, x) && \text{(by naturality of } \beta\text{)}\\
&= \alpha_D(H(f)(y), x) && \text{(since eval} \circ (\beta \circ \mathrm{id}_F) = \alpha\text{)}.
\end{aligned}$$

But the latter is exactly what defined $\hat{\alpha}$, so this shows that $\beta = \hat{\alpha}$ and completes our proof.

3. *NB. We sometimes use $\bullet$ to denote an arbitrary unnamed object of $\mathcal{C}$.*

We first describe the presheaf structure on Ω. Given an arrow $f\colon C \to D$ in $\mathcal{C}$, we define $\Omega(f)\colon \Omega(D) \to \Omega(C)$ by $S \mapsto \{g\colon \bullet \to C \mid f \circ g \in S\}$. We quickly verify that $\Omega(f)(S)$ is indeed a sieve: if $g \in \Omega(f)(S)$, then $f \circ g \in S$, so $f \circ g \circ h \in S$ for any h with suitable codomain as S is a sieve, meaning $g \circ h \in \Omega(f)(S)$ for such h, as desired. It is clear that $\Omega(\mathrm{id})(S) = S$, so we only need to check that Ω respects composition. Given $f\colon C \to D$ and $g\colon D \to E$ in $\mathcal{C}$, we have

$$\Omega(f)(\Omega(g)(S)) = \{h\colon \bullet \to D \mid f \circ h \in \Omega(g)(S)\} = \{h\colon \bullet \to C \mid g \circ f \circ h \in S\} = \Omega(g \circ f)(S),$$

as desired. Thus, Ω is indeed a presheaf.

Next, we define the natural transformation $\mathrm{true}\colon 1 \to \Omega$ by taking at every object C of $\mathcal{C}$ the unique element of $1(C)$ to the greatest sieve $\top_C$ on C, i.e. the set of all maps in $\mathcal{C}$ with codomain C. It is a useful observation that $\top_C$ is fully characterized by the fact that it contains id_C. Note that this assignment indeed defines a natural transformation, because if $f\colon C \to D$ is an arrow in $\mathcal{C}$, then $\Omega(f)(\top_C) = \{g\colon \bullet \to C \mid f \circ g \in \top_D\} = \top_C$, as $\top_D$ contains all morphisms to D, so the condition $f \circ g \in \top_D$ is satisfied for any map g to C.

If we are given a subobject of a presheaf B, then we may identify it with a *subpresheaf* of B. That is, a presheaf A such that $A(C) \subseteq B(C)$ for every object C and such that the action on morphisms of A is just the restriction of the action of B.

Now suppose that A is a subpresheaf of B. We construct a natural transformation $\chi\colon B \to \Omega$ as follows: for an object C, we define

$$\chi_C(x) \coloneqq \{g\colon C' \to C \text{ in } \mathcal{C} \mid B(g)(x) \in A(C')\}.$$

We must check that $\chi_C(x)$ is a sieve. So suppose that $g\colon C' \to C$ is in $\chi_C(x)$ and that $h\colon C'' \to C$ in $\mathcal{C}$. Then, $B(g \circ h)(x) = B(h)(B(g)(x)) \in A(C'')$, because $B(g)(x) \in A(C')$, so $g \circ h \in \chi_C(x)$ and $\chi_C(x)$ is seen to be a sieve.

Now, if $f\colon C \to D$ in $\mathcal{C}$ and $x \in B(D)$, then

$$\begin{aligned}
\chi_C(B(f)(x)) &= \{g\colon C' \to C \mid B(g)(B(f)(x)) \in A(C')\}\\
&= \{g\colon C' \to C \mid B(f \circ g)(x) \in A(C')\}\\
&= \{g\colon C' \to C \mid f \circ g \in \chi_D(x)\}\\
&= \Omega(f)(\chi_D(x)),
\end{aligned}$$

so χ is indeed a natural transformation.

We proceed by verifying that the diagram of natural transformations

$$\begin{tikzcd} A \arrow[r] \arrow[d, tail] & 1 \arrow[d, "\mathrm{true}"] \\ B \arrow[r, "\chi"'] & \Omega \end{tikzcd}$$

commutes. To this end, suppose C is an object of $\mathcal{C}$ and x is an element of $A(C)$. We have to show that $\chi_C(x)$ is the greatest sieve on C, which is equivalent to the condition that $\mathrm{id}_C \in \chi_C(x)$. But this condition holds, because $\chi_C(x) = \{g\colon C' \to C \mid B(g)(x) \in A(C')\}$ and $x \in A(C)$.

Next up, we prove that the square above is a pullback. Since limits are computed pointwise in $\mathcal{S}et^{\mathcal{C}^{\mathrm{op}}}$, it suffices to show that the square is a pullback in $\mathcal{S}et$ at every object C of $\mathcal{C}$. So suppose that we are given a set X with a function $\sigma\colon X \to B(C)$ such that $\chi_C(\sigma(x))$ is the greatest sieve on C for every element $x \in X$. Then $\mathrm{id}_C \in \chi_C(\sigma(x))$, meaning $\sigma(x) = B(\mathrm{id}_C)(\sigma(x)) \in A(C)$ for every $x \in X$, so σ factors uniquely through $A(C)$. Thus, the square is a pullback.

Finally, we show that χ is the unique natural transformation making the square into a pullback. To this end, suppose that χ' is another and let C be an arbitrary object of $\mathcal{C}$ and $x \in G(C)$. We show that $\chi_C(x)$ and $\chi'_C(x)$ are equal sieves.

Suppose $f \in \chi_C(x)$, i.e. $B(f)(x) \in A(C')$. Then $\chi'_{C'}(B(f)(x))$ is the greatest sieve on C, because $\chi'_{C'}$ makes the square commute. In particular, it contains id_C. But by naturality of χ' we have $\chi'_{C'}(B(f)(x)) = \Omega(f)(\chi'_C(x)) = \{g\colon \bullet \to C' \mid f \circ g \in \chi'_C(x)\} \ni \mathrm{id}_C$, which implies that $f \in \chi'_C(x)$, so $\chi_C(x) \subseteq \chi'_C(x)$.

Conversely, if $f \in \chi'_C(x)$, then $\mathrm{id}_{C'} \in \{g\colon \bullet \to C' \mid f \circ g \in \chi'_C(x)\} = \Omega(f)(\chi'_C(x)) = \chi'_{C'}(B(f)(x))$ by naturality of χ'. Thus, $\chi'_{C'}(B(f)(x))$ is the greatest sieve on C. Hence, by the assumption that χ' makes the square into a pullback, the element $B(f)(x)$ must be in $A(C')$. But this says exactly that $f \in \chi_C(x)$, completing the proof that $\chi_C(x) = \chi'_C(x)$ and establishing uniqueness of χ.