Topos Theory - Exercise Sheet 2

1. Prove that a distributive lattice also satisfies the *dual distributive law*:

$$a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$$

2. Recall that a *complement* of an element x of a lattice element a such that

$$x \vee a = 1 \qquad\qquad x \wedge a = 0$$

   Show that if X is a Heyting algebra and $x \in X$, then if a is a complement for x the it must be the negation of x. That is, show that $a = \neg x$ where $\neg x$ is defined as $x \Rightarrow 0$.

3. The purpose of this exercise will be to construct a map

$$\wedge : \Omega \times \Omega \to \Omega$$

   which "internalizes" the meet of two subterminal objects.

   (a) Show that if $f : X \rightarrowtail A$ and $g : X \rightarrowtail B$ are monomorphisms, then so is $(f, g) : X \rightarrowtail A \times B$.

   (b) Suppose

$$\begin{array}{ccc} A & \longrightarrow & B \\ \downarrow & & \downarrow \\ C & \longrightarrow & D \end{array} \qquad\qquad \begin{array}{ccc} X & \longrightarrow & Y \\ \downarrow & & \downarrow \\ Z & \longrightarrow & W \end{array}$$

   are two pullback squares. Show that the induced square

$$\begin{array}{ccc} A \times X & \longrightarrow & B \times Y \\ \downarrow & & \downarrow \\ C \times Z & \longrightarrow & D \times Y \end{array}$$

   is also a pullback.

   (c) Using the previous two steps, deduce that we have a map

$$\wedge : \Omega \times \Omega \to \Omega$$

such that for any pair of subterminal objects $U \rightarrowtail 1$ and $V \rightarrowtail 1$, the classifying diagram of the meet $U \times V \rightarrowtail 1$ may be factored as follows:

$$\begin{tikzcd}
U \times V \arrow[r] \arrow[d, tail] & 1 \arrow[r] \arrow[d] & 1 \arrow[d] \\
1 \arrow[r, "{(U,V)}"'] & \Omega \times \Omega \arrow[r, "\wedge"'] & \Omega
\end{tikzcd}$$

4. **Harder**: Can you construct maps

$$\begin{aligned}
\vee &: \Omega \times \Omega \to \Omega \\
\Rightarrow &: \Omega \times \Omega \to \Omega
\end{aligned}$$

which classify the other operations we have defined on subterminal objects?

5. (Descent for Monomorphisms) Let $\mathcal{E}$ be a topos and recall the category $\mathrm{Mono}(\mathcal{E})$ from the last exercise set.

   (a) Show that $\mathrm{Mono}(\mathcal{E})$ is equivalent to the slice category $\mathcal{E}/\Omega$.

   (b) Using the fact that $\mathcal{E}/\Omega$ is again a topos, conclude that $\mathrm{Mono}(\mathcal{E})$ has finite colimits.

   (c) Now consider a cube in which every vertical map is mono:

$$\begin{tikzcd}
A \arrow[rr] \arrow[dr] \arrow[dd, tail, "f"'] & & B \arrow[dr] \arrow[dd, tail, "g"' near start] & \\
& C \arrow[rr, crossing over] & & D \arrow[dd, tail, "k"] \\
E \arrow[rr] \arrow[dr] & & F \arrow[dr] & \\
& G \arrow[rr] \arrow[from=uu, tail, crossing over, "h"' near start] & & H
\end{tikzcd}$$

   Suppose that the back and left faces are pullback squares and that the top and bottom squares are pushout squares. Conclude the front and right faces are also pullback squares.