Topos Theory - Solutions to Exercise Sheet 1

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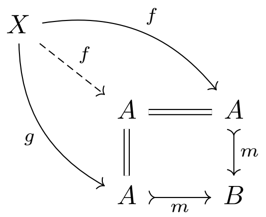
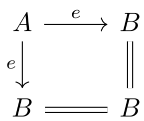
1. Suppose first that $m\colon A \rightarrowtail B$ is a monomorphism. We show that the given square is a pullback square, so let $f, g\colon X \to A$ be such that $m \circ f = m \circ g$. Because m is monic, we must have $f = g$, so that f (which equals g) is the unique dashed map making the diagram



commute. Hence, the given square is a pullback.

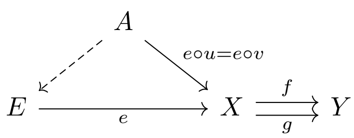
Conversely, suppose that the given square is a pullback and that we have $f, g\colon X \to A$ such that $m \circ f = m \circ g$. By the universal property of the pullback, there exists a (unique) $h\colon X \to A$ such that $f = \mathrm{id}_A \circ h = g$, so $f = g$, and m is seen to be a mono, as desired.

By duality, a map $e\colon A \to B$ is an epimorphism if and only if the square



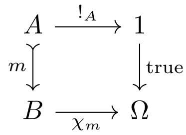
is a pushout.

2. Let $E \xrightarrow{e} X \overset{f}{\underset{g}{\rightrightarrows}} Y$ be an equalizer diagram. We show that e is a monomorphism. So suppose that we have $u, v\colon A \to E$ with $e \circ u = e \circ v$. Then $f \circ e \circ u = g \circ e \circ v$, so by the universal property of the equalizer, there exists a unique dashed map making the diagram

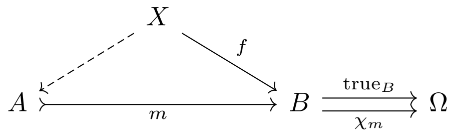


commute. Notice that both u and v make the diagram commute, so that $u = v$ follows by uniqueness of the dashed map, proving that e is a mono.

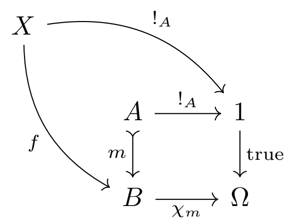
3. Recall that χ_m is such that the square



is a pullback. In particular, $\chi_m \circ m = \mathrm{true}_A \equiv \mathrm{true} \circ !_A = \mathrm{true} \circ !_A \circ m$, where the final equality holds, because 1 is the terminal object. Hence, the diagram in the exercise indeed commutes. It remains to prove the universal property of the equalizer. To this end, suppose that we are given a morphism $f\colon X \to B$ such that $\mathrm{true}_B \circ f = \chi_m \circ f$. We must find a unique dashed map making the diagram

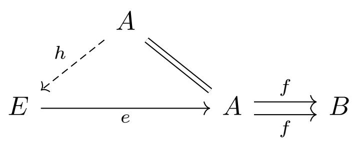


commute. Notice that the diagram

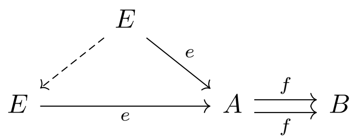


commutes. So by the universal property of the pullback, we get a unique $h\colon X \to A$ such that $m \circ h = f$ and $!_A \circ h = !_A$. The latter holds for any map into A, because 1 is the terminal object. Hence, there exists a unique $h\colon X \to A$ such that $m \circ h = f$, as desired.

4. By the universal property of the equalizer, there exists a unique $h\colon A \to E$ making the diagram



commute. It follows immediately from the diagram that $e \circ h = \mathrm{id}_A$, so it only remains to show that $h \circ e = \mathrm{id}_E$. To this end, note that by the universal property of the equalizer, there exists a unique dashed map making the diagram



commute. Obviously, we can take id_E for the dashed map, but $h \circ e$ also works, since $e \circ (h \circ e) = (e \circ h) \circ e = \mathrm{id}_A \circ e = e$. Hence, by uniqueness of the dashed map, we get the desired $h \circ e = \mathrm{id}_E$.

5. Spelling out the definitions, a terminal object in $M(\mathcal{E})$ is a monomorphism $t\colon S \rightarrowtail T$ of $\mathcal{E}$ such that for every monomorphism $m\colon A \rightarrowtail B$ of $\mathcal{E}$, there exists a unique pair of morphisms $u\colon A \to S$ and $v\colon B \to T$ making the square

$$\begin{array}{ccc} A & \xrightarrow{\;u\;} & S \\ {\scriptstyle m}\big\downarrow & & \big\downarrow{\scriptstyle t} \\ B & \xrightarrow[\;v\;]{} & T \end{array}$$

into a pullback. As shown in the lectures, it follows that S must be the terminal object of $\mathcal{E}$. Thus, any two maps into S with the same domain must be equal. Hence, we can replace "a unique pair…" above by just "a unique $v\colon B \to T$ …". But this says exactly that $t\colon S \rightarrowtail T$ is a subobject classifier.

6. (a) Suppose that $g, h\colon X \to A$ are such that $(\mathrm{id}_A, f) \circ g = (\mathrm{id}_A, f) \circ h$. We are to show that $g = h$, but this is straightforward, since $g = \pi_A \circ (\mathrm{id}_A, f) \circ g = \pi_A \circ (\mathrm{id}_A, f) \circ h = h$.

   (b) Suppose that $f'\colon A \to B$ is such that $G_f = G_{f'}$ as subobjects of $A \times B$. This means that we have an isomorphism $\varphi\colon A \cong A$ satisfying $(\mathrm{id}_A, f) \circ \varphi = (\mathrm{id}_A, f')$. Then notice that $(\mathrm{id}_A, f) \circ \varphi = (\varphi, f \circ \varphi)$, from which it follows by post-composing with π_A that $\varphi = \mathrm{id}_A$. Hence, $f = \pi_B \circ (\mathrm{id}_A, f) = \pi_B \circ (\mathrm{id}_A, f) \circ \varphi = \pi_B \circ (\mathrm{id}_A, f') = f'$, as we wished to show.