Stabilization of Homotopy Limits

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Abstract

This thesis presents a stable filtration of spaces which arise as homotopy limits. The filtration is inspired by Goodwillie's *Calculus of Functors* and can be regarded as an interpolation between the natural map

$$\Sigma^\infty \operatorname*{holim}_{\mathcal{C}} F \to \operatorname*{holim}_{\mathcal{C}} \Sigma^\infty F$$

for a given functor $F : \mathcal{C} \to \mathcal{S}*$ where $\mathcal{C}$ is a small category and $\mathcal{S}_*$ the category of pointed spaces. Our results can be regarded as a generalization of the Goodwillie tower for mapping spaces, as described in the work of G. Arone.

After constructing the tower, we provide a description of the layers and prove a convergence result to the effect that if the functor F takes values in spaces more highly connected that the dimension of the nerve of $\mathcal{C}$, then the tower converges.

For my parents, Ken and Susan.

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Contents

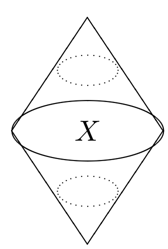
Chapter 1

Introduction

This thesis is concerned with exhibiting a certain filtration of the stable homotopy type of spaces which arise as homotopy limits. The filtration produced is inspired by Goodwillie's Calculus of Functors, and can be viewed as a generalization of the stable filtration of mapping spaces described in [1]. In this introduction, we provide some motivation and describe the results obtained in later chapters.

Stable Homotopy Theory

A fundamental construction in homotopy theory is the *suspension*, ΣX, of a topological space X, formed by "gluing two cones" to X as in the following digram.



More formally, ΣX is the quotient of $X \times [0, 1]$ obtained by collapsing to a point each of the subspaces $X \times \{0\}$ and $X \times \{1\}$. Given a continuous map $f : X \to Y$, one can easily see that there is an induced map $\Sigma f : \Sigma X \to \Sigma Y$ by simply applying f levelwise, so that we may regard $\Sigma : \mathcal{S} \to \mathcal{S}$ as a *functor* on the category $\mathcal{S}$ of spaces.

A major motivation for studying the suspension functor is that it tends to make spaces more tractable. One example of this phenomenon, of which we will have quite a lot more to say in Chapter 5, is the following: given spaces X and Y with a chosen basepoint in each, we have a homotopy equivalence

$$\Sigma(X \times Y) \simeq \Sigma X \vee \Sigma Y \vee \Sigma(X \wedge Y)$$

In other words, up to homotopy, the suspension of a cartesian product of two spaces "splits" into a wedge sum of its constituent components. (Recall that the wedge $\vee$ of two pointed spaces is the space obtained by gluing together their basepoints, and that the smash product $X \wedge Y$ can be defined as $X \wedge Y = X \times Y / X \vee Y$.)

If we proceed under the assumption that suspending a space X may help us break it into simpler pieces, then we could try to simplify X even further by *repeatedly* suspending it. Thus one is led naturally to consider the sequence of spaces

$$X,\ \Sigma X,\ \Sigma^2 X,\ \ldots$$

where we inductively define $\Sigma^n X = \Sigma(\Sigma^{n-1} X)$. One thinks of the above sequence as capturing all the essential information in the space X which remains after an arbitrarily high number of suspensions.

It is convenient to organize such sequences into a category $\mathcal{S}p$, called the category of *spectra*, by making the following definition: a *spectrum* is a sequence of pointed spaces $\{X_i\}_{i=0}^{\infty}$ together with structure maps

$$\alpha_i : \Sigma X_i \to X_{i+1}$$

To each pointed space X, the sequence $\{\Sigma^i X\}_{i=0}^{\infty}$ described above (where we have taken $\alpha_i = \mathrm{id}_{\Sigma^{i+1} X}$) forms a spectrum called the *suspension spectrum* of X which is commonly denoted $\Sigma^{\infty} X$.

It turns out that this definition is more general than might appear at first glance in the sense that there are many spectra which do not arise as the suspension spectrum of a space X. For example, the *Brown representability theorem* asserts that every (generalized) cohomology theory $E^*(-) : \mathcal{S} \to \mathcal{G}r\mathcal{A}b$ (where $\mathcal{G}r\mathcal{A}b$ is the category of graded abelian groups) determines a spectrum $E \in \mathcal{S}p$. As an example, ordinary singular cohomology $H^*(-, \mathbb{Z})$ is represented by the Eilenberg-MacLane spectrum $H\mathbb{Z}$ defined by

$$H\mathbb{Z} = \{K(\mathbb{Z}, i)\}_{i=0}^{\infty}$$

where $K(\mathbb{Z}, i)$ is the i-th Eilenberg-MacLance space associated to the abelian group $\mathbb{Z}$. Moreover, the definition is set up to ensure that the converse also holds: every spectrum determines a cohomology (and homology) theory on the category of spaces. For this reason, the category $\mathcal{S}p$ of spectra has many properties in common with purely algebraic categories such as the category of chain complexes over a ring R, and serves as a kind of intermediate ground between topology and pure algebra. *Stable homotopy theory* is concerned with the properties of the category $\mathcal{S}p$.

Associating to each space X its suspension spectrum determines a functor

$$\Sigma^{\infty} : \mathcal{S} \to \mathcal{S}p$$

so that we can view the ordinary category of spaces as reflected in the stable category. While the above discussion has made the case that the study of $\Sigma^{\infty} X$ is often more tractable than the study of X itself, there are still major difficulties in practice. Taking the case $X = S^0$, the 0-sphere, we find that understanding the spectrum $\Sigma^{\infty} S^0$ is equivalent to calculating the stable homotopy groups of spheres, perhaps the most important unsolved problem of stable homotopy theory today.

On the other hand, there are cases where are great deal more can be said. A classic example is that of the James model $J(X)$ for the loop space of a suspension. [13]. Milnor [16] showed that after a single suspension, this space splits into a wedge of smash products of X. In particular, this implies that stably we have

$$\Sigma^\infty \Omega \Sigma X \simeq \bigvee_{i=0}^{\infty} \Sigma^\infty X^{\wedge i}$$

where $\Omega \Sigma X = \mathrm{Map}(S^1, \Sigma X)$ is the space of based loops in ΣX. Observe that this theorem can be phrased as a statement about the suspension spectrum $\Sigma^\infty \mathrm{Map}(S^1, \Sigma X)$ of a *mapping space*. Moreover, the resulting spectrum has a natural *filtration* by the sub-spectra

$$\bigvee_{i=1}^{n} \Sigma^\infty X^{\wedge i}$$

for increasing n. We will be concerned with similar stable filtrations in what follows.

The Goodwillie Calculus

A common way to try and understand a complicated space X (or spectrum, as we shall see) is to *filter* X by finding some natural family of spaces $\{X_i\}$ which have either a map to or from the space X in question. As an example, when X is a CW-complex (or simplicial set), we have the *skeletal filtration* defined by letting X_i be the subspace of X determined by all cells (respectively simplices) of dimension less than or equal to i. The subspaces X_i determine a family of inclusions

$$X_i \hookrightarrow X$$

and moreover

$$X \cong \mathrm{colim}(X_0 \hookrightarrow X_1 \hookrightarrow X_2 \hookrightarrow \cdots)$$

Often one can recover important informaion about X (such as, say, the homology or cohomology of X) from information about the X_i and the successive quotients X_i/X_{i-1}. Observe that each of these spaces appears as the *cofiber* of the inclusion $X_{i-1} \hookrightarrow X_i$, and that in the case at hand, each is a bouquet of spheres.

Dually, a space X has a Postnikov tower, or *coskeletal filtration*, formed by spaces PX_i defined by the property that $\pi_k(PX_i) = 0$ for $k > i$ and $\pi_k(PX_i) = \pi_k(X)$ for $k < i$. Each PX_i comes equipped with a natural fibration $PX_i \to X$ and we have

$$X \cong \lim(PX_0 \leftarrow PX_1 \leftarrow PX_2 \leftarrow \cdots)$$

We find from caclulating in the long exact sequence of homotopy groups induced by the fibrations $PX_i \to PX_{i-1}$ that the *fiber* of this map is an Eilenberg-MacLane space. We can again try to recover information about X from knowledge of its Postnikov sections PX_i and the fibers just described.

Often the process of recovering information about the orginal space X takes the form of a *spectral sequence*, which can be viewed as an algebraic machine taking as input the information about the filtration, and producing as output the required information about X. Subtle variations in the way one filters a given object X can lead to spectral sequences with very different properties, some more manageable than others.

The *Goodwillie Calculus*, developed in the series of papers [9], [10], [11], can be thought of as a structured way of generating such filtrations. The setup, however, is slightly more general: instead of filtering a fixed space X, we consider some homotopy functor F and obtain a filtration of *all* the spaces (or spectra) $F(X)$ simultaneously. In the case of relevance to this thesis, we deal with functors $F : \mathcal{S}_* \to \mathcal{S}p$, where $\mathcal{S}_*$ is the category of *pointed* spaces and $\mathcal{S}p$ is the category of spectra. The Goodwillie Calculus then produces for us a filtration of the spectrum $F(X)$ given by a tower of fibrations

$$\begin{tikzcd}
 & \vdots \arrow[d] \\
 & P_nF(X) \arrow[d] \\
 & P_{n-1}F(X) \arrow[d] \\
 & \vdots \arrow[d] \\
F(X) \arrow[r] \arrow[uuur] \arrow[uur] & P_1F(X)
\end{tikzcd}$$

valid *for each space* X. One thinks of the functors $P_nF : \mathcal{S}_* \to \mathcal{S}p$ so produced as "polynomial approximations" to the given functor F, in analogy with the Taylor approximations of a real-valued function f. We will refer to the above tower as either the *Goodwillie* or *Taylor* tower for the functor F. It is shown in [11], that the fiber of the map

$$P_nF(X) \to P_{n-1}F(X)$$

takes the form

$$(\partial_n F \wedge X^{\wedge n})_{h\Sigma_n}$$

where $\partial_n F$ is a spectrum with an action of the n-th symmetric group Σ_n called the *n-th Goodwillie derivative* of F. For a given homotopy functor F, one is interested in understanding the spectra $\partial_n F$ for each n, the approximating functors P_nF, and in understanding what conditions on the space X will ensure that we have an equivalence

$$F(X) \simeq \varprojlim(P_1F(X) \leftarrow P_2F(X) \leftarrow \cdots)$$

When the above equivalence holds, one says that the tower *converges* at X. Determining the spaces X for which this happens is analogous to finding the "radius of convergence" of the Taylor series of some real-valued function f.

There are certain functors F for which answers to these questions are known. For example, [1] gives a complete desciption of the Goodwillie Tower for the functor $F = \Sigma^\infty \operatorname{Map}(K, X)$, with K some fixed finite dimensional complex. To describe the result, we first introduce some notation. Let $\mathcal{E}^{\mathrm{op}}$ denote the opposite of the category of non-empty finite sets and surjective maps and $\mathcal{E}_n^{\mathrm{op}}$ the subcategory of finite sets of cardinality at most n. Then each pointed space X determines a functor $\mathcal{E}^{\mathrm{op}} \to \mathcal{S}_*$ with

$$p \mapsto X^{\wedge p}$$

and by restriction, a functor $\mathcal{E}_n^{\mathrm{op}} \to \mathcal{S}_*$ for each n. For fixed X and K, we let $\operatorname{Nat}_{\mathcal{E}_n^{\mathrm{op}}}(\Sigma^\infty K^{\wedge p}, \Sigma^\infty X^{\wedge p})$ denote the spectrum of natural transformations between the associated functors, which can be described as the *end*

$$\int_{\mathcal{E}_n^{\mathrm{op}}} \operatorname{Map}_{\mathcal{S}p}(\Sigma^\infty K^{\wedge p}, \Sigma^\infty X^{\wedge p})$$

Write $c(X)$ for the connectivity of a pointed space X. Then the following theorem is proved in [1]

Theorem 1.0.1 (Arone)**.** *For each n, we have a weak equivalence*

$$P_n \Sigma^\infty \operatorname{Map}(K, X) \xrightarrow{\sim} \operatorname{Nat}_{\mathcal{E}_n^{\mathrm{op}}}(\Sigma^\infty K^{\wedge p}, \Sigma^\infty X^{\wedge p})$$

Moreover, the Goodwillie tower converges for all spaces X which satisfy

$$c(X) > \dim K$$

We will obtain this result as a corollary of the convergence properties of our stabilization tower for homotopy limits in Chapter 6.

For more applications of the Goodwillie Calculus, the reader may wish to consult, for example, [2], [14], [20], [19].

A Stable Filtration for Homotopy Limits

Given a small category $\mathcal{C}$ and a functor $F : \mathcal{C} \to \mathcal{S}_*$, one can construct a space called the *homotopy limit* of F which repairs a certain deficiency of the ordinary limit: namely that it is not homotopy invariant. We will give precise definitions in the next chapter, but the heuristic discussion which follows will serve well enough for the introduction.

Recall that to each category $\mathcal{C}$ we can associate a space $\mathcal{N}(\mathcal{C})$ (really, a simplicial set, though the distinction will not concern us here) called the *nerve* of $\mathcal{C}$. Then our functor $F : \mathcal{C} \to \mathcal{S}_*$ determines a space, the *homotopy colimit* of F, together with a natural projection $\pi : \operatorname{hocolim}_{\mathcal{C}} F \to \mathcal{N}(\mathcal{C})$. One imagines constructing the homotopy colimit by taking the disjoint union of all $F(x)$ as x runs over the objects of $\mathcal{C}$, and gluing these spaces together using the topology of $\mathcal{N}(\mathcal{C})$. For example, when F is the terminal functor, i.e. constant at a point, then $\operatorname{hocolim}_{\mathcal{C}} F = \mathcal{N}(\mathcal{C})$. The projection π is defined so that it satisfies $\pi^{-1}(x) = F(x)$ for all $x \in \mathcal{C}$.

An intuitive definition of the homotopy limit, then, is as the space of sections of the projection π. That is

$$\operatorname*{holim}_{\mathcal{C}} F = \Gamma \left(\begin{tikzcd} \operatorname{hocolim}_{\mathcal{C}} F \arrow[d, shift left, "\pi"] \\ \mathcal{N}(\mathcal{C}) \arrow[u, dashed, bend left, shift left] \end{tikzcd} \right)$$

where we have used Γ to denote the indicated space of sections.

One can check that if the functor $F : \mathcal{C} \to \mathcal{S}_*$ is constant at some space X, that is, $F(x) = X$ for all $x \in \mathcal{C}$, then $\operatorname{hocolim}_{\mathcal{C}} F \simeq \mathcal{N}(\mathcal{C}) \times X$. One then expects that

$$\operatorname*{holim}_{\mathcal{C}} F \simeq \operatorname{Map}(\mathcal{N}(\mathcal{C}), X)$$

which is indeed the case. We see then from this discussion that one should regard the homotopy limit of a functor $F : \mathcal{C} \to \mathcal{S}_*$ as a kind of space of sections, or generalized mapping space, of which the ordinary mapping space is a special case.

Allowing the functor F to vary, we may regard the homotopy limit as a functor

$$\operatorname{holim} : \mathcal{S}_*^{\mathcal{C}} \to \mathcal{S}_*$$

One could say, then, that the goal of the current work is to exhibit the Goodwillie tower for the composite functor

$$\Sigma^\infty \operatorname{holim} : \mathcal{S}_*^{\mathcal{C}} \to \mathcal{S}p$$

in such a way that it agrees with the calculation of [1] when restricted to *constant* functors.

One caveat before we continue: while this point of view provides motivation for the current work, and even though we will in fact be able to recover the results described in the previous section, we will stop short of referring to our filtration as the Goodwillie tower for the homotopy limit functor. The reason is that such a claim would require a fully developed theory of the Goodwillie Calculus on diagram categories. While such a theory almost certainly exists, it is not yet fully treated in the literature, and we will not have the means to explore it here. In particular, we do not prove any result about the excisive properties of our tower, which would certainly be a requirement for any filtration claiming to be the Goodwillie Tower. We choose instead to refer to our filtration as the *stabilization tower* for homotopy limits with the hope that we can remove this restriction in future work.

Let us see, then, how one might obtain the desired filtration of $\Sigma^\infty \operatorname{holim}_{\mathcal{C}} F$ for some diagram $F : \mathcal{C} \to \mathcal{S}_*$ of pointed spaces. As a first approximation, one might consider the spectrum $\operatorname{holim}_{\mathcal{C}} \Sigma^\infty F$ obtained by first stabilizing the values of the functor F pointwise. Indeed, one can easily check that there is a natural map

$$\Sigma^\infty \operatorname*{holim}_{\mathcal{C}} F \to \operatorname*{holim}_{\mathcal{C}} \Sigma^\infty F$$

But this map in rarely an equivalence. Compare this, however, with the case $n = 1$ in Theorem 1.0.1. (This is often referred to as the *linear approximation.*) This map takes the form

$$\Sigma^\infty \operatorname{Map}(K, X) \to \operatorname{Map}(\Sigma^\infty K, \Sigma^\infty X)$$

So we might expect the natural map above to play the role of the linear approximation in the stabilization tower for homotopy limits, and this is indeed the case.

To obtain the higher order approximations, that is, the higher levels in the filtration, we consider the role played by the category $\mathcal{E}_n^{\text{op}}$ of non-empty finite sets of cardinality at most n and surjections. In Section 4.1, we will create a category $\mathcal{C}^{(n)}$ built as a kind of wreath product of $\mathcal{C}$ and $\mathcal{E}_n^{\text{op}}$ by employing a certain categorical operation called the Grothendieck construction, described in detail in Chapter 3. We construct a natural extension $F^{(n)} : \mathcal{C}^{(n)} \to \mathcal{S}_*$ of our original functor to this new category, and this construction determines a tower

$$\begin{tikzcd}
 & \vdots \arrow[d] \\
 & \operatorname*{holim}_{\mathcal{C}^{(n)}} \Sigma^\infty F^{(n)} \arrow[d] \\
 & \operatorname*{holim}_{\mathcal{C}^{(n-1)}} \Sigma^\infty F^{(n-1)} \arrow[d] \\
 & \vdots \arrow[d] \\
\Sigma^\infty \operatorname*{holim}_{\mathcal{C}} F \arrow[r] \arrow[uuur] \arrow[uur] & \operatorname*{holim}_{\mathcal{C}^{(1)}} \Sigma^\infty F^{(1)}
\end{tikzcd}$$

which yields the desired filtration.

Of primary interest in any tower type filtration are the *layers* of the tower. That is, the homotopy fibers

$$\operatorname*{holim}_{\mathcal{C}^{(n)}} \Sigma^\infty F^{(n)} \to \operatorname*{holim}_{\mathcal{C}^{(n-1)}} \Sigma^\infty F^{(n-1)}$$

for varying n. We present two descriptions, both of which follow from our groundwork on *fibered* categories in Chapter 3. The main idea is to “cone off” the parts of each homotopy limit arising from lower levels in the tower. It turns out that in order to excise as much as possible from each level, we need to employ a kind of subdivision process which we explore in Section 3.3.

We carry out our plan as follows. In Chapter 2, we recall some of the basic constructions necessary for the remainder of the discussion. We discuss homotopy limits, homotopy Kan extensions, cosimplicial spaces and totalization. Then in Chapter 3 we introduce the notion of fibered and opfibered categories, showing how they are related to the Grothendieck construction alluded to above. We also explore how to manipulate homotopy limits indexed by such categories, as this will be a fundamental tool both in exploring the tower itself and investigating its convergence properties. In Chapter 4 we construct the tower as outlined above, and provide our aforementioned descriptions of the *layers*. Chapter 5 contains a description of the tower when the category $\mathcal{C}$ is discrete and finite, and we show that we recover the classical stable splitting of a product in this case. Finally in Chapter 6 we prove a result about the convergence of the tower under certain hypothesis on the indexing category $\mathcal{C}$ and the values of the functor F. Our main result says that if $\mathcal{N}(\mathcal{C})$ is finite, and if for each $x \in \mathcal{C}$ we have

$$c(F(x)) \geq \dim \mathcal{N}(\mathcal{C}/x)$$

the the stabilization tower for $\operatorname{holim}_{\mathcal{C}} F$ converges.

Chapter 2

Background

In this chapter we recall some of the basic ideas fundamental to this thesis.

2.1 Notation

Some various categories we will be using:

$$
\begin{aligned}
\Delta &= \text{Finite Totally Ordered Sets} \\
\mathcal{S} &= \text{Simplicial Sets} \\
\mathcal{S}_* &= \text{Pointed Simplicial Sets} \\
\mathcal{S}p &= \text{Spectra} \\
\mathcal{E} &= \text{Finite Sets and Surjections} \\
\mathcal{E}_n &= \text{Finite Sets and Surjections of cardinality at most } n
\end{aligned}
$$

2.2 Homotopy Limits

The standard reference for this material is [4]. See also the paper [17] for a slightly more modern treatment.

The starting point for the introduction of homotopy limits is the following observation: if $F : \mathcal{C} \to \mathcal{S}_*$ and $G : \mathcal{C} \to \mathcal{S}_*$ are two $\mathcal{C}$-diagrams of pointed simplicial sets, and $\alpha : F \to G$ is a natural transformation which is an object-wise weak equivalence, then the natural map

$$\varprojlim F \to \varprojlim G$$

need not be a weak equivalence. In a sense, this should not be much of a surprise. Homotopy theory takes the view that two spaces should be considered equivalent when there is a map between them inducing an isomorphism on all homotopy groups. But, of course, such a map need not be an isomorphism of simplicial sets. As the ordinary limit is defined without reference to these special maps, we should not expect it to respect this extra structure.

One can easily repair this difficulty using the language of model categories (see [12] or [7]). Recall that a category $\mathcal{M}$ is said to have all $\mathcal{C}$-indexed limits if the diagonal functor $D : \mathcal{M} \to \mathcal{M}^{\mathcal{C}}$ which sends an object of $\mathcal{M}$ to the *constant* functor at that object has a right adjoint. If $\mathcal{M}$ is a closed model category then the theory allows us to construct a *derived adjunction* between $\mathrm{Ho}(\mathcal{M})$ and $\mathrm{Ho}(\mathcal{M}^{\mathcal{C}})$. The right adjoint of this adjunction is called the *homotopy limit* functor. As the category of pointed simplicial sets carries the structure of a closed model category, we can form, for any small category $\mathcal{C}$ and any functor $F : \mathcal{C} \to \mathcal{S}_*$ an object

$$\operatorname*{holim}_{\mathcal{C}} F \in \mathcal{S}_*$$

called the homotopy limit of F.

Various explicit models for this construction can be given, all of which result in weakly equivalent spaces. We will assume from here on that our functor takes values in *fibrant* simplicial sets, as can always be arranged by composing F with a fibrant replacement functor (for example, Kan's Ex^{∞}.) In this case, one explicit model is given as the *end*

$$\operatorname*{holim}_{\mathcal{C}} F = \int_{\mathcal{C}} \mathrm{Map}_*(\mathcal{N}(\mathcal{C}/x)_+, F(x))$$

where $\mathcal{N}$ denotes the nerve functor. A reference for ends, and in particular the universal property they enjoy, is [15]. We will introduce a second model shortly when we consider cosimplicial spaces.

The following extremely useful result was first proved in [4]. We quote the more general result of [12, Theorem 19.6.7] which applies to any model category.

Proposition 2.2.1. *Let $G : \mathcal{D} \to \mathcal{C}$ be a functor and $F : \mathcal{C} \to \mathcal{M}$ an objectwise fibrant diagram in a model category $\mathcal{M}$. Suppose that $\mathcal{N}(G/x)$ is contractible for every $x \in \mathcal{C}$. Then G induces a natural weak equivalence*

$$\operatorname*{holim}_{\mathcal{C}} F \xrightarrow{\sim} \operatorname*{holim}_{\mathcal{D}} FG$$

We will encounter the situation of the preceding proposition many times in this thesis, after verifying its hypotheses, we will often just say that an equivalence "follows by confinality."

Example 2.2.1. Let $F : \mathcal{C} \to \mathcal{S}$ be a functor which is *constant*. Say $F(x) = X$ for all $x \in \mathcal{C}$. Then we have

$$\begin{aligned}
\operatorname*{holim}_{\mathcal{C}} F &= \int_{x\in\mathcal{C}} \mathrm{Map}_*(\mathcal{N}(\mathcal{C}/x)_+, F(x)) \\
&= \int_{x\in\mathcal{C}} \mathrm{Map}(\mathcal{N}(\mathcal{C}/x), X) \\
&= \mathrm{Map}(\int^{x\in\mathcal{C}} \mathcal{N}(\mathcal{C}/x), X) \\
&= \mathrm{Map}(\mathcal{N}(\mathcal{C}), X)
\end{aligned}$$

as claimed in the introduction. Here we have used elementary properties of *end calculus* as described in [15], together with the fact that the nerve of a category is the colimit of the nerve of its slice categories.

2.3 Homotopy Kan Extensions

Let $\mathcal{C}$ be a small category, and $F : \mathcal{C} \to \mathcal{M}$ a $\mathcal{C}$-diagram in $\mathcal{M}$. Given a functor $G : \mathcal{C} \to \mathcal{D}$, we can ask whether there is any natural extension of F to a functor $\mathcal{D} \to \mathcal{M}$. That is, we wish to find a functor filling in the dotted arrow in the diagram

$$\begin{tikzcd} \mathcal{C} \arrow[rr, "G"] \arrow[dr, "F"'] & & \mathcal{D} \arrow[dl, dashed] \\ & \mathcal{M} & \end{tikzcd}$$

If $\mathcal{M}$ has small limits, we can construct such an extension, called the *right Kan extension* of F along G, and denoted $\operatorname{Ran}_G F$ by setting

$$(\operatorname{Ran}_G F)(d) = \lim_{c \in G/d} F(c)$$

One checks that this indeed gives a well defined functor, and moreover that we have an adjunction $G^* \vdash \operatorname{Ran}_G$ where G^* denotes the restriction functor $G^* : \mathcal{M}^{\mathcal{D}} \to \mathcal{M}^{\mathcal{C}}$ given by precomposition with G.. (Again, see [15].) Observe that when $\mathcal{D} = \{*\}$ is the terminal category, the functor obtained by right Kan extension along the unique functor $\mathcal{C} \to \{*\}$ just picks out the limit of F in $\mathcal{M}$.

When the base category $\mathcal{M}$ is a cofibrantly generated closed model category, there is an exactly analogous situation. Again denoting the restriction functor $\mathcal{M}^{\mathcal{D}} \to \mathcal{M}^{\mathcal{C}}$ by G^*, we define a functor $G_* : \mathcal{M}^{\mathcal{C}} \to \mathcal{M}^{\mathcal{D}}$ by setting

$$G_*(F)(d) = \operatorname*{holim}_{c \in G/d} F(c)$$

The resulting functor is not strictly a right adjoint for G^*, but becomes one after passing to the homotopy category [5, Theorem 6.11]. That is to say, there is an induced adjunction

$$\operatorname{Ho}(\mathcal{M}^{\mathcal{C}}) \underset{G^*}{\overset{G_*}{\rightleftarrows}} \operatorname{Ho}(\mathcal{M}^{\mathcal{D}})$$

We deduce, in particular

Proposition 2.3.1. *Let $\mathcal{M}$ be a cofibrantly generated model category. Suppose we have functors*

$$\mathcal{C} \xrightarrow{G} \mathcal{D} \xrightarrow{H} \mathcal{K}$$

Then for any functor $F : \mathcal{C} \to \mathcal{M}$, we have a natural weak equivalence

$$(H_* \circ G_*)(F) \simeq (H \circ G)_*(F)$$

Proof. This follows from the fact that adjoint functors compose, and that if two objects are naturally isomorphic in $\mathrm{Ho}(\mathcal{M}^{\mathcal{K}})$ then they are naturally weakly equivalent in $\mathcal{M}^{\mathcal{K}}$. □

Corollary 2.3.2. *Let $\mathcal{M}$ be a cofibrantly generated model category $G : \mathcal{C} \to \mathcal{D}$ a functor between small categories, and $F : \mathcal{C} \to \mathcal{M}$ a $\mathcal{C}$ diagram in $\mathcal{M}$. Then there is a natural weak equivalence*

$$\operatorname*{holim}_{\mathcal{C}} F \simeq \operatorname*{holim}_{\mathcal{D}} G_*(F)$$

Proof. Take $\mathcal{K} = \{*\}$ in Proposition 2.3.1. □

We remark that the theory of Kan extensions between diagram categories for a fixed cofibrantly generated closed model category $\mathcal{M}$ can be given an elegant formulation using the theory of *derivators*. In particular, the reader can find complete proofs of the above assertions in the papers [6] and [5].

2.4 Cosimplicial Spaces

Let Δ denote the category of finite ordinals and order preserving maps. A functor $X : \Delta \to \mathcal{S}$ is called a *cosimplicial space*. We write $X^p = X([p])$ for the functors value at the object $[p] \in \Delta$. Given a cosimplicial space X, we associate to X a space $\mathrm{Tot}\, X$ called its *totalization* as follows. Observe that we have a canonical cosimplicial space $\Delta : \Delta \to \mathcal{S}$ defined by sending $[p] \mapsto \Delta^p$, the standard p-simplex. (In fact, the functor Δ is just the Yoneda embedding.) Then we can define

$$\mathrm{Tot}\, X = \int_\Delta \mathrm{Map}(\Delta^p, X^p)$$

That is, $\mathrm{Tot}\, X$ is just the space of natural transformations from the standard cosimplicial space to X. This construction is exactly dual to the notion of the *realization* of a simplicial space.

The category Δ has an increasning sequence of subcategories $\Delta^{\leq n}$ obtained by considering only the finite ordinals up to cardinality $n+1$. One can easily check that

$$\Delta \cong \mathrm{colim}\left(\Delta^{\leq 0} \to \Delta^{\leq 1} \to \Delta^{\leq 2} \to \cdots\right)$$

For each n, we can construct a kind of truncated totalization, denoted $\mathrm{Tot}_n\, X$ by setting

$$\mathrm{Tot}_n\, X = \int_\Delta \mathrm{Map}(\mathrm{sk}_n\, \Delta^p, X([p]))$$

where $\mathrm{sk}_n : \mathcal{S} \to \mathcal{S}$ denotes the n-th skeleton functor. One easily deduces that $\mathrm{Tot}\, X$ is isomorphic to the inverse limit of the tower

$$\begin{tikzcd}
 & \vdots \arrow[d] \\
 & \mathrm{Tot}_n\, X \arrow[d] \\
 & \mathrm{Tot}_{n-1}\, X \arrow[d] \\
 & \vdots \arrow[d] \\
\mathrm{Tot}\, X \arrow[r] \arrow[uur] \arrow[uuur] & \mathrm{Tot}_0\, X
\end{tikzcd}$$

We refer to this as the Tot *tower* for X.

In the case when the cosimplicial space X is *Reedy fibrant*, that is, a fibrant object in the Reedy model structure on cosimplicial spaces, (see [12], or [4]), the the totalization of

X has another description. In this case [12, Theorem 18.7.4] shows that we have

$$\operatorname{Tot} X \simeq \operatorname*{holim}_{\Delta} X$$

and similarly

$$\operatorname{Tot}_n X \simeq \operatorname*{holim}_{\Delta^{\leq n}} X$$

Cosimplicial spaces are an indispensible tool in the study of mapping spaces, which are, as we have pointed out special cases of homotopy limits. The reason is that given any functor $F : \mathcal{C} \to \mathcal{S}_*$, we can form a cosimplicial space, called the *cosimplicial replacement* of F as follows.

Write $\mathcal{C}_p$ for the set of p-simplices of the category $\mathcal{C}$. That is, $\mathcal{C}_p = \operatorname{Hom}_{\mathcal{C}at}([p], \mathcal{C})$, considered as a set. An arbitrary element of $\mathcal{C}_p$ is just a chain of p composable morphisms in $\mathcal{C}$. We will write $\overrightarrow{x} = x_0 \to \cdots \to x_p$ for an arbitrary element of $\mathcal{C}_p$. Now define

$$X_F^p = \prod_{\substack{x_0 \to \cdots \to x_p \\ \in \mathcal{C}_p}} F(x_p)$$

Then $X_F : \Delta \to \mathcal{S}_*$ is a cosimplicial space, where the value of the functor X_F on morphisms of Δ is given by pre-composition of chains. Moreover, the following is proved in [4, Lemma 5.2].

Proposition 2.4.1. *If F is an objectwise fibrant $\mathcal{C}$-diagram of pointed simplical sets, then we have a natural weak equivalence*

$$\operatorname*{holim}_{\mathcal{C}} F \simeq \operatorname{Tot} X_F$$

where $\operatorname{holim}_{\mathcal{C}} F$ *is defined as in Section 2.2.*

In this thesis, we will be exclusively concerned with cosimplicial spaces arising as the cosimplicial replacement of a functor $F : \mathcal{C} \to \mathcal{S}_*$, and all such diagrams are Reedy fibrant.

Thus, in light of the above discussion, we have equivalences

$$\operatorname*{holim}_{\mathcal{C}} F \simeq \operatorname{Tot} X_F \simeq \operatorname*{holim}_{\Delta} X_F$$

and we will freely use whichever model is most convenient in a given situation.

2.5 Cubical Diagrams

We write $\underline{n} = \{1, 2, \ldots n\}$ for the finite set with n elements and $\mathcal{P}_{\underline{n}}$ for the partially ordered set of subsets of $\underline{n}$ regarded as a small category. More generally, for any finite set S, we write $\mathcal{P}_S$ for the category of subsets of S.

Definition 2.5.1. A *cubical diagram* is a functor

$$\mathcal{X} : \mathcal{P}_{\underline{n}} \to \mathcal{S}_*$$

for some $\underline{n}$. We occasionally refer to such a functor as an *n-cube* or n-cubical diagram.

Let $S \subseteq T \subseteq \underline{n}$. The full subcategory of $\mathcal{P}_{\underline{n}}$ determined by the objects $\{U \subseteq \underline{n} \mid S \subseteq U \subseteq T\}$ is isomorphic to $\mathcal{P}_{T \setminus S}$, and hence the restriction of a given cubical diagram $\mathcal{X} : \mathcal{P}_{\underline{n}} \to \mathcal{S}_*$ to this subcategory is again a cubical diagram, which we denote by $\partial_S^T \mathcal{X}$. If either $S = \emptyset$ or $T = \underline{n}$, we will omit it from the notation. The case $S = \{x\}$, $T = \underline{n} \setminus \{y\}$ where $x, y \in \underline{n}$ occurs often enough to merit the notation $\partial_x^y \mathcal{X}$.

For a given $x \in \underline{n}$ we have a canonical map of $(n-1)$-cubes

$$\partial^x \mathcal{X} \to \partial_x \mathcal{X}$$

for any $x \in \underline{n}$. Given our notational conventions, the domain of this natural transformation is simply the restriction of $\mathcal{X}$ to all subsets U for which $x \notin U$, while the codomain is the restriction of $\mathcal{X}$ to all V such that $x \in V$. The components of the natural transformation are given by evaluating $\mathcal{X}$ on the inclusions $U \hookrightarrow U \cup \{x\}$ for all U such that $x \notin U$. By taking the homotopy fiber of each of the components of this natural transformaion, we obtain an $(n-1)$ cube $\operatorname{hofib}_x \mathcal{X}$. Continuing inductively, we finally arrive at a 0-cube, that is, a space, which we denote by $\operatorname{tfib} \mathcal{X}$ and refer to as the *total fiber* of $\mathcal{X}$.

An alternate description of the total fiber, and one which makes it clear that its homotopy type does not depend on the order in which the elements are chosen above, is given as follows. Let $\mathcal{P}^+_{\underline{n}}$ denote the category of *non-empty* subsets of $\underline{n}$. We have a canonical map

$$a(\mathcal{X}) : \mathcal{X}(\emptyset) \to \operatorname*{holim}_{\mathcal{P}^+_{\underline{n}}} \mathcal{X}$$

One finds the following characterization of the total fiber proved in [10].

Proposition 2.5.1. *We have a natural equivalence*

$$\operatorname{tfib} \mathcal{X} \simeq \operatorname{hofib} a(\mathcal{X})$$

Note that the above discussion has a dual form. That is, one may succesively take the homotopy cofiber of the maps $\partial^x \mathcal{X} \to \partial_x \mathcal{X}$ and arrive at a space called the *total cofiber* of $\mathcal{X}$. We have, in this case, a canonical map

$$b(\mathcal{X}) : \operatorname*{hocolim}_{\mathcal{P}^-_{\underline{n}}} \mathcal{X} \to \mathcal{X}(\underline{n})$$

where $\mathcal{P}^-_{\underline{n}}$ is the subcategory determined by omitting the terminal object $\underline{n}$. The analog of Proposition 2.5.1 is given by

Proposition 2.5.2. *We have a natural equivalence*

$$\operatorname{tcofib} \mathcal{X} \simeq \operatorname{hocofib} b(\mathcal{X})$$

A cubical diagram $\mathcal{X}$ is called *cartesian* if the map $a(\mathcal{X})$ is an equivalence and *k-cartesian* if it is k-connected. Dually, $\mathcal{X}$ is *cocartesian* if $b(\mathcal{X})$ is an equivalence and *k-cocartesian* if it is k-connected. Goodwillie's *generalized Blakers-Massey* theorem provides a relationship between these two notions. We will make use of the following form in our proof of convergence below.

Theorem 2.5.3. *Let $\mathcal{X}$ be an S-cube with $|S| = n \geq 1$. Suppose that*

1. *for each $T \neq \emptyset$, the T-cube $\partial_{S-T}\mathcal{X}$ is $\kappa(T)$-cartesian and*

2. *$\kappa(U) \leq \kappa(T)$ whenever $U \subset T$.*

Then $\mathcal{X}$ is d-cocartesian where d is the minimum of $n - 1 + \sum_\alpha \kappa(T_\alpha)$ over all partitions $\{T_\alpha\}$ of S by non-empty sets.

Our interest in cubical diagrams arises from the following connection with the Tot tower of a cosimplicial space. Fix some $[p] \in \Delta$ with $p > 0$. Then one easily checks that the category $\Delta_{\mathcal{M}}/[p]$ where $\Delta_{\mathcal{M}}$ denotes the subcategory of injective order preserving maps is isomorphic to $\mathcal{P}^+_{\underline{p+1}}$, the isomorphism given by sending an injective map $\mu : [q] \to [p]$ to the subset $\operatorname{im}\mu \subseteq [p]$.

Observe that the category $\Delta_{\mathcal{M}}/[p]$ comes equipped with a canonical projection π_p : $\Delta_{\mathcal{M}}/[p] \to \Delta$, and hence, given a cosimplicial space X, the composition

$$\Delta_{\mathcal{M}}/[p] \xrightarrow{\pi_p} \Delta \xrightarrow{X} \mathcal{S}_*$$

can be considered as a functor on $\mathcal{P}^+_{\underline{p+1}}$, that is, as a cubical diagram missing its initial object. A proof of the following fact can be found in [18].

Proposition 2.5.4. *We have an equivalence*

$$\operatorname*{holim}_{\Delta_{\mathcal{M}}/[p]} X \circ \pi_p \simeq \operatorname{Tot}_n \widetilde{X}$$

where $\widetilde{X}$ denotes a fibrant replacement for X in the model structure on cosimplicial spaces.

Chapter 3

Fibered and Opfibered Categories

In this chapter we introduce the notions of a fibered and opfibered categories. Roughly, these are categories arising from what is know as the Grothendieck construction, which can be regarded as an analog of the homotopy colimit in $\mathcal{C}at$, the category of small categories. We investigate the properties of homotopy limits indexed by diagrams arising from this construction and provide some examples.

3.1 Opfibered Categories and the Grothendieck Construction

Definition 3.1.1. Let $\pi : \mathcal{E} \to \mathcal{B}$ be a functor between small categories. If $f : a \to b$ is an arrow of $\mathcal{B}$, and $x \in \pi^{-1}(a)$ is an object of $\mathcal{E}$, we say that an arrow $u : x \to y$ is a *opcartesian arrow* for f and x if $\pi(u) = f$ and for any arrow $v : x \to z$ and any arrow $h : b \to \pi(z)$ for which $h \circ f = \pi(v)$, there is a unique arrow $w : y \to z$ in $\mathcal{E}$ such that $w \circ u = v$ and $\pi(w) = h$.

This definition is perhaps best understood visually.

$$
\begin{tikzcd}
\mathcal{E} \arrow[d, dashed, "\pi"] \\
\mathcal{B}
\end{tikzcd}
\qquad
\begin{tikzcd}
 & & z \arrow[dd, dashed] \\
x \arrow[urr, bend left, "v"] \arrow[r, "u"] \arrow[dd, dashed] & y \arrow[dd, dashed] & \\
 & & \pi(z) \\
a \arrow[urr, bend left, "\pi(v)"] \arrow[r, "f"'] & b \arrow[ur, "h"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
 & & z \arrow[dd, dashed] \\
x \arrow[urr, bend left, "v"] \arrow[r, "u"] \arrow[dd, dashed] & y \arrow[dd, dashed] \arrow[ur, "\exists ! w"'] & \\
 & & \pi(z) \\
a \arrow[urr, bend left, "\pi(v)"] \arrow[r, "f"'] & b \arrow[ur, "h"'] &
\end{tikzcd}
$$

Definition 3.1.2. A functor $\pi : \mathcal{E} \to \mathcal{B}$ between small categories is called an *opfibration* if there exists an opcartesian arrow for every pair (f, x) where $f : a \to b$ is an arrow of $\mathcal{B}$ and $x \in \pi^{-1}(a) \subseteq \mathcal{E}$.

The following fundamental construction is a major source of opfibered categories.

Definition 3.1.3. Let $\mathcal{C}$ be a small category and $F : \mathcal{C} \to \mathcal{C}at$ a $\mathcal{C}$-diagram of small categories. The *Grothendieck construction* on F, denoted $\int F$ is the category defined as follows: the objects of $\int F$ are pairs (a, x) where a is an object of $\mathcal{C}$ and x is an object of $F(a)$. A morphism $(a, x) \to (b, y)$ is a pair (f, h) where $f : a \to b$ is a morphism of $\mathcal{C}$ and $h : F(f)(x) \to y$ is a morphism of $F(b)$. Given two morphism $(f, h) : (a, x) \to (b, y)$ and $(g, k) : (b, y) \to (c, z)$. composition in the category $\int F$ is defined by

$$(f, h) \circ (g, k) = (g \circ f, k \circ F(g)(h))$$

That this composition is well defined is guaranteed by the functoriality of F.

Observe that there is a natural projection $\pi : \int F \to \mathcal{C}$ defined by $\pi(a, x) = a$. In fact, we have

Proposition 3.1.1. *Let $F : \mathcal{C} \to \mathcal{C}at$ be a $\mathcal{C}$-diagram of small categories. Then the natural projection*

$$\pi : \int F \to \mathcal{C}$$

is a opfibration.

Proof. Let $f : a \to b$ be a morphism of $\mathcal{C}$ and let $(a, x) \in \pi^{-1}(a)$ (so that $x \in F(a)$). Then the pair $(F(f)(x), \mathrm{id}_{F(f)(x)})$ is an opcartesian arrow for f. $\square$

Example 3.1.1. Let $L : \mathcal{C} \to \mathcal{D}$ be a functor and fix some $d \in \mathcal{D}$. Consider the functor

$$\mathrm{Hom}_{\mathcal{D}}(d, L(-)) : \mathcal{C} \to \mathcal{S}et$$

and regard $\mathcal{S}et$ as the full subcategory of $\mathcal{C}at$ consisting of the discrete categories.

By definition, an object of the category $\int \mathrm{Hom}_{\mathcal{D}}(d, L(-))$ is a pair (c, x) where c is an object of $\mathcal{C}$ and $x \in \mathrm{Hom}_{\mathcal{D}}(d, L(-))$. That is, $x : d \to L(c)$ is morphism in $\mathcal{D}$. We denote this category by d/L and refer to it as a *coslice* category. When $L = \mathrm{id}$, one calls this the category of objects *under* d.

Example 3.1.2. Consider the diagram $C_{\mathcal{D}} : \mathcal{C} \to \mathcal{C}at$ which is *constant* at $\mathcal{D}$. That is, $C_{\mathcal{D}}(c) = \mathcal{D}$ for every object $c \in \mathcal{C}$ and $C_{\mathcal{D}}(f) = \mathrm{id}_{\mathcal{D}}$ for every morphism $f \in \mathcal{C}$. In this case, an object of $\int C_{\mathcal{D}}$ is a pair (c, d) where $c \in \mathcal{D}$ and $d \in \mathcal{D} = C_{\mathcal{D}}(c)$. A morphism is a pair $(f, g) : (c, d) \to (c', d')$ where $f : c \to c'$ is a morphism of $\mathcal{C}$ and $g : F(f)(d) = d \to d'$ is a morphism of $\mathcal{D}$. It follows that $\int C_{\mathcal{D}} \cong \mathcal{C} \times \mathcal{D}$ and we learn that for any two categories $\mathcal{C}$ and $\mathcal{D}$, the projection map

$$\mathcal{C} \times \mathcal{D} \to \mathcal{C}$$

is an opfibration.

3.2 Fibered Categories

Dual to the notion of an opfibered category is that of a fibered category. We record the definition here for completeness.

Definition 3.2.1. Let $\pi : \mathcal{E} \to \mathcal{B}$ be a functor between small categories. If $f : a \to b$ is an arrow of $\mathcal{B}$, and $y \in \pi^{-1}(b)$ is an object of $\mathcal{E}$, we say that an arrow $u : x \to y$ is a *cartesian arrow* for f and y if $\pi(u) = f$ and for any arrow $v : z \to y$ and any arrow $h : \pi(z) \to a$ for which $f \circ h = \pi(v)$, there is a unique arrow $w : z \to x$ in $\mathcal{E}$ such that $w \circ u = v$ and $\pi(w) = h$.

Remark 3.2.1. This definition is perhaps most easily understood by viewing it as a kind of "relative pullback" property as pictured in the following diagrams

$$
\begin{tikzcd}
\mathcal{E} \arrow[d, dashed, "\pi"] \\
\mathcal{B}
\end{tikzcd}
\qquad
\begin{tikzcd}
z \arrow[rrd, bend left, "v"] \arrow[dd, dashed] & & \\
& x \arrow[r, "u"] \arrow[dd, dashed] & y \arrow[dd, dashed] \\
\pi(z) \arrow[rd, "h"'] \arrow[rrd, bend left, "\pi(v)"] & & \\
& a \arrow[r, "f"'] & b
\end{tikzcd}
\qquad
\begin{tikzcd}
z \arrow[rrd, bend left, "v"] \arrow[dd, dashed] \arrow[rd, "\exists ! w"'] & & \\
& x \arrow[r, "u"] \arrow[dd, dashed] & y \arrow[dd, dashed] \\
\pi(z) \arrow[rd, "h"'] \arrow[rrd, bend left, "\pi(v)"] & & \\
& a \arrow[r, "f"'] & b
\end{tikzcd}
$$

The dotted arrows in these diagrams indicate application of the functor π.

Definition 3.2.2. A functor $\pi : \mathcal{E} \to \mathcal{B}$ between small categories is called a *fibration* if there exists a cartesian arrow for every pair (f, y) where $f : a \to b$ is an arrow of $\mathcal{B}$ and $y \in \pi^{-1}(b) \subseteq \mathcal{E}$.

We record for later use the following lemma, which is just a consequence of the duality of the definitions involved.

Lemma 3.2.1. *Let $\pi : \mathcal{E} \to \mathcal{B}$ be an opfibration. Then $\pi^{\mathrm{op}} : \mathcal{E}^{\mathrm{op}} \to \mathcal{B}^{\mathrm{op}}$ is a fibration.*

As a consequence, given a functor $F : \mathcal{C}^{\mathrm{op}} \to \mathcal{C}at$, we can create a category fibered over $\mathcal{C}$, denoted ∇F, by setting

$$\nabla F = \left(\int F^{\mathrm{op}} \right)^{\mathrm{op}}$$

where F^{op} denotes the functor defined by $F^{\mathrm{op}}(c) = F(c)^{\mathrm{op}}$ for all $c \in \mathcal{C}$.

Our goal is to show that homotopy limits over a fibered category $\mathcal{E}$ decompose in a simple fashion. A first step in this direction is the following:

Proposition 3.2.2. *Let $\pi : \mathcal{E} \to \mathcal{B}$ be a fibration. Then the inclusion functor*

$$i : \pi^{-1}(a) \to a/\pi$$

admits a right adjoint for every object $a \in \mathcal{B}$.

Proof. We must first define a functor $j : a/\pi \to \pi^{-1}(a)$. An object of a/π is a pair (y, f) where $y \in \mathcal{E}$ and $f : a \to \pi(y)$ is a morphism of $\mathcal{B}$. Then since π is a fibration, there exists a cartesian arrow $u : x \to y$ for which $\pi(u) = f$. In particular $x \in \pi^{-1}(a)$ and we may set $j(y, f) = x$.

We are now to check that we have a natural isomorphism

$$\mathrm{Hom}_{\pi^{-1}}(z, j(y, f)) \cong \mathrm{Hom}_{a/\pi}(i(z), (y, f))$$

Now, by composition with the map u, a map $w : z \to x$ in $\pi^{-1}(a)$ determines a map $(z, \mathrm{id}_z) \to (y, f)$ in a/π. Moreover, the fact that u is cartesian says that every such map arises this way. □

We recall the following standard fact about adjoint functors.

Lemma 3.2.3. *Let $\lambda : \mathcal{C} \to \mathcal{D}$ be a functor which admits a right adjoint $\rho : \mathcal{D} \to \mathcal{C}$. Then for every object $d \in \mathcal{D}$, the category λ/d has an terminal object.*

Proof. Observe that the counit $\lambda(\rho(d)) \to d$ makes $\rho(d)$ into an object of λ/d. Moreover, the adjunction triangle

$$\begin{tikzcd}
\lambda(\rho(d)) \arrow[r] & d \\
\lambda(c) \arrow[u] \arrow[ur] &
\end{tikzcd}$$

shows that $\rho(d)$ is a terminal object. □

Remark 3.2.2. An adjunction between small categories induces a simplicial homotopy equivalence between their nerves. Thus the previous proposition can be viewed as saying that the strict fiber and the homotopy fiber of a fibration are equivalent. This is one justification for the use of the term *fibration*.

We now acheive the desired decomposition.

Proposition 3.2.4. *Let $\pi : \mathcal{E} \to \mathcal{B}$ be a fibration and let $F : \mathcal{E} \to \mathcal{S}_*$ be an $\mathcal{E}$-diagram of pointed simplicial sets. Then the homotopy right Kan extension of F along π, $\pi_*(F)$, is naturally weakly equivalent to the functor defined by*

$$b \mapsto \operatorname*{holim}_{\pi^{-1}(b)} F_b$$

where F_b denotes the restriction of F to the fiber lying over $b \in \mathcal{B}$.

Proof. From the definition, we have

$$\pi_*(F)(b) = \operatorname*{holim}_{\pi/b} F$$

but Lemma 3.2.3 and Proposition 3.2.2 show that the inclusion $\pi^{-1}(b) \to \pi/b$ is cofinal, so the result follows from Proposition 2.2.1. □

Corollary 3.2.5. *We have*

$$\operatorname*{holim}_{\mathcal{E}} F \simeq \operatorname*{holim}_{\mathcal{B}} \operatorname*{holim}_{\pi^{-1}(b)} F_b$$

Proof. This now follows from Corollary 2.3.2. □

The corollary tells us that when computing homotopy limits over fibered categories, we may first calculate the homotopy limit of each strict fiber, and then compute the homotopy limit of the resulting spaces. We now apply this result to prove some simple lemmas which will be useful later.

Write $\mathbb{N}$ for the partially ordered set of natural numbers regarded as a category. In other words, the category pictured as

$$0 \to 1 \to 2 \to 3 \to \cdots$$

Suppose a given category $\mathcal{C}$ is filtered by a family of subcategories $\mathcal{C}_i$ in the sense that $\mathcal{C}$ is the colimit of the diagram

$$\mathcal{C}_0 \hookrightarrow \mathcal{C}_1 \hookrightarrow \mathcal{C}_2 \hookrightarrow \cdots$$

in $\mathcal{C}at$. Regard the above diagram as a functor $K : \mathbb{N} \to \mathcal{C}at$ and set

$$\mathcal{C}_\infty = \nabla K$$

One can think of $\mathcal{C}_\infty$ as a kind of reverse mapping telescope. It is fibered over $\mathbb{N}^{\mathrm{op}}$ via the natural projection $\pi : \mathcal{C}_\infty \to \mathbb{N}^{\mathrm{op}}$, and the fiber over an element $i \in \mathbb{N}^{\mathrm{op}}$ is the subcategory $\mathcal{C}_i$. Explicitly, an object of $\mathcal{C}_\infty$ is a pair (i, x) where $i \in \mathbb{N}^{\mathrm{op}}$ and $x \in \mathcal{C}_i$. We remark that for $i < j$ there is a canonical map $(j, x) \to (i, x)$.

We picture this category as follows:

$$\begin{tikzcd}
\mathcal{C}_\infty \arrow[d, dashed, "\pi"] & & \mathcal{C}_0 \arrow[d, dashed] & \mathcal{C}_1 \arrow[l] \arrow[d, dashed] & \mathcal{C}_2 \arrow[l] \arrow[d, dashed] & \cdots \arrow[l] \\
\mathbb{N}^{\mathrm{op}} & & 0 & 1 \arrow[l] & 2 \arrow[l] & \cdots \arrow[l]
\end{tikzcd}$$

Observe that in this case, we also have a natural functor $\gamma : \mathcal{C}_\infty \to \mathcal{C}$ defined by $\gamma(i, x) = x \in \mathcal{C}$.

Lemma 3.2.6. *The functor $\gamma : \mathcal{C}_\infty \to \mathcal{C}$ is cofinal.*

Proof. Let $x \in \mathcal{C}$. We must show that $\mathcal{N}(\gamma/x)$ is contractible. Let n be the least natural number for which $x \in \mathcal{C}_n$. Define an endofunctor $\rho_n : \gamma/x \to \gamma/x$ as follows.

$$\rho_n(i, y) = \begin{cases} (n, y) & i < n \\ (i, y) & i \geq n \end{cases}$$

Observe that there is a natural transformation $\rho_n \to \mathrm{id}_{\gamma/x}$ whose component at (i,y) is the identity for $i \geq n$ and the canonical map $(n,y) \to (i,y)$ for $i < n$. It follows that the nerve of γ/x is simplicially homotopy equivalent to the image of ρ_n. But (n,x) is a terminal object of this subcategory.

Corollary 3.2.7. *If $F : \mathcal{C} \to \mathcal{S}_*$ is a $\mathcal{C}$ diagram of pointed simplicial sets, then* $\mathrm{holim}_{\mathcal{C}} F$ *is naturally weakly equivalent to the homotopy inverse limit of its restrictions to the $\mathcal{C}_i$. That is,*

$$\operatorname*{holim}_{\mathcal{C}} F \simeq \operatorname*{holim}_{\mathbb{N}^{\mathrm{op}}} \left(\operatorname*{holim}_{\mathcal{C}_0} F_0 \leftarrow \operatorname*{holim}_{\mathcal{C}_1} F_1 \leftarrow \operatorname*{holim}_{\mathcal{C}_2} F_2 \leftarrow \cdots \right)$$

Proof. Apply Corollary 3.2.5. □

Example 3.2.1. Take $\mathcal{C} = \Delta$ and filter Δ by the subcategories $\Delta^{\leq n}$ of finite totally ordered sets with at most $n+1$ elements. We have seen that for a cosimplicial space $X : \Delta \to \mathcal{S}_*$ we have an equivalence

$$\mathrm{Tot}_n \simeq \operatorname*{holim}_{\Delta^{\leq n}} X$$

So Corollary 3.2.7 gives a simple proof of the fact that the totalization of a cosimplicial space is the homotopy inverse limit of its Tot tower.

□

3.3 The Twisted Arrow Category

For any category $\mathcal{C}$, we have the functor

$$\mathrm{Hom}_{\mathcal{C}}(-,-) : \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathcal{S}et$$

Regarding $\mathcal{S}et$ as a full subcategory of $\mathcal{C}at$, we may consider the category $T(\mathcal{C}) = \int \mathrm{Hom}_{\mathcal{C}}$. We refer to this category as the *twisted arrow category* of $\mathcal{C}$. Explicitly, the objects are the morphisms $f : a \to b$ of $\mathcal{C}$ and a morphism $f \to f'$ in $T(\mathcal{C})$ is a commutative square

$$
\begin{tikzcd}
a \arrow[r, "f"] & b \arrow[d] \\
a' \arrow[u] \arrow[r, "f'"'] & b'
\end{tikzcd}
$$

in $\mathcal{C}$. The category $T(\mathcal{C})$ can be viewed as a kind of subdivision of the original category. In fact, the nerve of $T(\mathcal{C})$ agrees with the *ordinal subdivision* of [8].

The composition

$$\pi : T(\mathcal{C}) \to \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathcal{C}$$

sends an arrow $f : a \to b$ to its codomain $b \in \mathcal{C}$. One easily checks that for $b \in \mathcal{C}$, we have $\pi^{-1}(b) \cong (\mathcal{C}/x)^{\mathrm{op}}$. As this category has an initial object, we have in particular

Lemma 3.3.1. *For any small category $\mathcal{C}$ and any $\mathcal{C}$-diagram $F : \mathcal{C} \to \mathcal{M}$ in a simplicial model category, the restriction map*

$$\operatorname*{holim}_{\mathcal{C}} F \to \operatorname*{holim}_{T(\mathcal{C})} F \circ \pi$$

is an equivalence.

Proof. This follows from Proposition 2.2.1. $\square$

The construction of the twisted arrow category is functorial in the category $\mathcal{C}$ and hence determines an endofunctor $T : \mathcal{C}at \to \mathcal{C}at$. We now show that this functor carries opfibrations to fibrations.

Proposition 3.3.2. *Suppose $\pi : \mathcal{E} \to \mathcal{B}$ is an opfibration. Then the induced functor*

$$T(\pi) : T(\mathcal{E}) \to T(\mathcal{B})$$

is a fibration.

Proof. Consider an arrow $(h, k) : f \to f'$ in $T(\mathcal{B})$, depicted as in the diagram

$$\begin{tikzcd} a & a' \arrow[l, "h"'] \arrow[d, "f'"] \\ b \arrow[u, leftarrow, "f"] \arrow[r, "k"'] & b' \end{tikzcd}$$

in $\mathcal{B}$. If $u' : x' \to y'$ is an an arrow of $\mathcal{E}$ for which $\pi(u') = f'$, then we may regard this as an object of $T(\mathcal{E})$ which lies over f'. Hence we must show that there is a cartesian arrow for u' lying over the morphism (h, k).

As $\pi(x') = a'$, and since the functor π is an opfibration, we can find an opcartesian arrow for the pair (x', h), that is, an arrow $r : x' \to x$ for some x with $\pi(x) = a$. But now x lies over a, and we can select an opcartesian arrow over f, say $u : x \to y$. Finally, we select some $s : y \to z$ with $\pi(z) = b'$.

The arrows r, u, s are all opcartesian, hence so is the composite

$$x' \xrightarrow{r} x \xrightarrow{u} y \xrightarrow{s} z$$

Then by definition, we obtain a unique arrow $t : z \to y'$ such that $t \circ s \circ u \circ r = u'$.. One easily checks that the morphism $(r, t \circ s)$ depicted as

$$\begin{tikzcd} x & x' \arrow[l, "r"'] \arrow[d, "u'"] \\ y \arrow[u, leftarrow, "u"] \arrow[r, "t \circ s"'] & y' \end{tikzcd}$$

is the cartesian lift required. $\square$

Chapter 4

Construction of the Stabilization Tower for Homotopy Limits

4.1 The Category $\mathcal{C}^{(\infty)}$

Let us write $\mathcal{F}$ for the category of finite sets. Let $\mathcal{A}$ be a category with finite products. Fixing some $A \in \mathcal{A}$, we observe that there is a functor

$$P_A : \mathcal{F}^{\mathrm{op}} \to \mathcal{A}$$

defined by setting

$$P_A(S) = \prod_{s \in S} A$$

for a finite set S. Given a morphism $f : S \to T$ in $\mathcal{F}$, the morphism $P_A(T) \to P_A(S)$ is defined as follows. An element $s \in S$ determines a projection

$$\pi_{f(s)} : P_A(T) = \prod_{t \in T} A \to A$$

and the collection of these projections as s runs over the elements of S determines a unique map

$$P_A(f) : \prod_{t \in T} A \xrightarrow{\prod \pi_{f(s)}} \prod_{s \in S} A$$

One says that any category $\mathcal{A}$ with finite products is *cotensored* over finite sets.

Observing that the category $\mathcal{C}at$ of small categories has products, we now specialize to the case $\mathcal{A} = \mathcal{C}at$ and write $\mathcal{E}$ the category of *non-empty* finite sets of the form $\underline{n} = \{1, \ldots, n\}$ and surjections. We thus obtain, by composition with the natural inclusion, a functor

$$E_{\mathcal{C}} : \mathcal{E}^{\mathrm{op}} \to \mathcal{F}^{\mathrm{op}} \xrightarrow{P_{\mathcal{C}}} \mathcal{C}at$$

for every category $\mathcal{C} \in \mathcal{C}at$. Explicitly, we have $E_{\mathcal{C}}(\underline{n}) = \mathcal{C}^n$ for a finite set $\underline{n}$.

Definition 4.1.1. We set

$$\mathcal{C}^{(\infty)} = \int E_{\mathcal{C}}$$

Unraveling the definitions, we find that an object of $\mathcal{C}^{(\infty)}$ is a pair $(\underline{n}, x)$ where $x \in \mathcal{C}^n$. It will be convenient to use the notation $x_{\underline{n}} = (x_1, \ldots, x_n)$ for such an object, as it is completely determined by its n components. A morphism $f : (y_1, \ldots, y_k) \to (x_1, \ldots, x_n)$ consists of a surjection $\sigma : \underline{n} \to \underline{k}$ together with an $\underline{n}$-indexed family of morphisms $\{f_i : y_{\sigma(i)} \to x_i\}_{i=1}^n$ with each $f_i \in \mathcal{C}$.

Remark 4.1.1. Fix some $x_{\underline{n}} = (x_1, \ldots, x_n) \in \mathcal{C}^{(\infty)}$. Then from the above description of the morphisms of $\mathcal{C}^{(\infty)}$, we see that there is a natural projection

$$\mathcal{C}^{(\infty)}/(x_1, \ldots, x_n) \to \mathcal{C}/x_1 \times \cdots \times \mathcal{C}/x_n$$

which sends the morphism defined by the family $\{f_i\}_{i=1}^n$ and the surjection σ to the object of the codomain category which is given by the tuple $(f_1, \ldots, f_n)$.

4.2 Constructing the Tower

Let $F : \mathcal{C} \to \mathcal{S}_*$ be a $\mathcal{C}$-diagram of pointed, fibrant simplicial sets. We construct a functor $F^{(\infty)} : \mathcal{C}^{(\infty)} \to \mathcal{S}_*$ as follows. For $(x_1, \ldots, x_n) \in \mathcal{C}^{(\infty)}$, we set

$$F^{(\infty)}(x_1, \ldots, x_n) = F(x_1) \wedge \cdots \wedge F(x_n)$$

Suppose we have a map $(\sigma, \{f_i\}_{i=1}^n) : (x_1, \ldots, x_m) \to (y_1, \ldots, y_n)$ in the notation above. Our object is to define a map

$$F(x_1) \wedge \cdots \wedge F(x_m) \to F(y_1) \wedge \cdots \wedge F(y_n)$$

Note that for $1 \leq i \leq n$ we have the map $F(f_i) : F(x_{\sigma(i)}) \to F(y_i)$. Then we can define

$$(a_1, \ldots, a_m) \mapsto (F(f_1)(a_{\sigma(1)}), \ldots, F(f_n)(a_{\sigma(n)}))$$

for $(a_1, \ldots, a_m) \in F(x_1) \times \cdots \times F(x_m)$. The fact that these maps descend to the smash product follows from the fact that we can view the smash product as the Cartesian product modulo the *thick* wedge, meaning that an element $(a_1, \ldots, a_m)$ is equivalent to the basepoint if and only if at least one of the a_i is a basepoint. But then the fact that the f_i are based maps guarantees that the image of this element is also a basepoint, and hence the above map induces a well defined map on the smash product.

We now show

Lemma 4.2.1. *There is a natural map*

$$\operatorname*{holim}_{\mathcal{C}} F \to \operatorname*{holim}_{\mathcal{C}^{(\infty)}} F^{(\infty)}$$

Proof. The assumption that F is objectwise fibrant allows us to explicitly construct each homotopy limit as an end. We have the formulas

$$\operatorname*{holim}_{\mathcal{C}} F = \int_{\mathcal{C}} \mathrm{Map}_*(\mathcal{N}(\mathcal{C}/x), F(x))$$
$$\operatorname*{holim}_{\mathcal{C}^{(\infty)}} F^{(\infty)} = \int_{\mathcal{C}^{(\infty)}} \mathrm{Map}_*(\mathcal{N}(\mathcal{C}^{(\infty)}/x_{\underline{n}}), F(x_{\underline{n}}))$$

Recall that to construct a map into an end, it is sufficient to construct maps to each component which satisfy an approriate commutative diagram. In the case at hand, we must construct maps

$$\omega_{x_{\underline{n}}} : \operatorname*{holim}_{\mathcal{C}} F \to \mathrm{Map}_*(\mathcal{N}(\mathcal{C}^{(\infty)}/x_{\underline{n}}), F^{(\infty)}(x_{\underline{n}}))$$

for each object $x_{\underline{n}} = (x_1, \cdots, x_n)$ of $\mathcal{C}^{(\infty)}$ which make the diagram

$$\begin{tikzcd}
\operatorname{holim}_{\mathcal{C}} F \arrow[r] \arrow[d] & \mathrm{Map}_*(\mathcal{N}(\mathcal{C}^{(\infty)}/x_{\underline{n}}), F(x_{\underline{n}})) \arrow[d] \\
\mathrm{Map}_*(\mathcal{N}(\mathcal{C}^{(\infty)}/x_{\underline{m}}), F(x_{\underline{m}})) \arrow[r] & \mathrm{Map}_*(\mathcal{N}(\mathcal{C}^{(\infty)}/x_{\underline{n}}), F(x_{\underline{m}}))
\end{tikzcd}$$

commute for every map $x_{\underline{n}} \to x_{\underline{m}}$ in $\mathcal{C}^{(\infty)}$. Observe that the projection

$$\epsilon_i : \mathcal{C}^{(\infty)}/x_{\underline{n}} \to \prod_{i=1}^{n} \mathcal{C}/x_i \to \mathcal{C}/x_i$$

of Remark 4.1.1 allows us to form the composite

$$\omega_i : \operatorname*{holim}_{\mathcal{C}} F \to \mathrm{Map}_*(\mathcal{N}(\mathcal{C}/x_i), F(x_i)) \xrightarrow{\mathcal{N}(\epsilon_i)^*} \mathrm{Map}_*(\mathcal{N}(\mathcal{C}^{(\infty)}/x_{\underline{n}}), F(x_i))$$

Taking the product over i and passing to the smash product finally yields the desired map

$$\omega_{x_{\underline{n}}} : \operatorname*{holim}_{\mathcal{C}} F \xrightarrow{\prod \omega_i} \mathrm{Map}_*(\mathcal{N}(\mathcal{C}^{(\infty)}/x_{\underline{n}}), \prod F(x_i)) \to \mathrm{Map}_*(\mathcal{N}(\mathcal{C}^{(\infty)}/x_{\underline{n}}), F^{(\infty)}(x_{\underline{n}}))$$

It is straigtforward, if tedious, to check that the required square commutes. $\square$

The category $\mathcal{C}^{(\infty)}$ is filtered by the full subcategories $\mathcal{C}^{(n)}$ consisting of objects $x_{\underline{k}} = (x_1, \ldots, x_k)$ with $1 \leq k \leq n$. Composing the map of the previous lemma with the restriction of homotopy limits induced by the inclusion $\mathcal{C}^{(n)} \to \mathcal{C}^{(\infty)}$, we obtain a map

$$\operatorname*{holim}_{\mathcal{C}} F \to \operatorname*{holim}_{\mathcal{C}^{(n)}} F_n$$

for every n. Moreover, we conclude from Corollary 3.2.7 that $\operatorname{holim}_{\mathcal{C}^{(\infty)}} F^{(\infty)}$ is the inverse limit of these restrictions.

Recall that for any space X, we have a natural map $X \to \Omega^\infty \Sigma^\infty X$, the unit of the adjuction $\Sigma^\infty \vdash \Omega^\infty$ between spaces and spectra. From naturality, it follows that we have an induced map

$$\operatorname*{holim}_{\mathcal{C}^{(n)}} F_n \to \operatorname*{holim}_{\mathcal{C}^{(n)}} \Omega^\infty \Sigma^\infty F_n$$

Recall that the functor Ω^∞ preserves homotopy limits, so that we have an equivalence $\operatorname{holim}_{\mathcal{C}^{(n)}} \Omega^\infty \Sigma^\infty F_n \simeq \Omega^\infty \operatorname{holim}_{\mathcal{C}^{(n)}} \Sigma^\infty F_n$. Hence composing with the map of the previous paragraph and taking a transpose, we obtain a map

$$\Sigma^\infty \operatorname*{holim}_{\mathcal{C}} F \to \operatorname*{holim}_{\mathcal{C}^{(n)}} \Sigma^\infty F_n$$

We have now proved

Theorem 4.2.2. *Let $F : \mathcal{C} \to \mathcal{S}_*$ be a $\mathcal{C}$-diagram of pointed simplicial sets. Then $\Sigma^\infty \operatorname{holim}_{\mathcal{C}} F$ maps naturally to the tower*

$$\begin{tikzcd} & \vdots \arrow[d] \\ & \operatorname*{holim}_{\mathcal{C}^{(n)}} \Sigma^\infty F^{(n)} \arrow[d] \\ & \operatorname*{holim}_{\mathcal{C}^{(n-1)}} \Sigma^\infty F^{(n-1)} \arrow[d] \\ & \vdots \arrow[d] \\ \Sigma^\infty \operatorname*{holim}_{\mathcal{C}} F \arrow[r] \arrow[uuur] \arrow[uur] & \operatorname*{holim}_{\mathcal{C}^{(1)}} \Sigma^\infty F^{(1)} \end{tikzcd}$$

We refer to this as the *stabilization tower* for the homotopy limit of F.

Before examining the layers of the tower, let us consider two special cases.

Lemma 4.2.3. *Let $\mathcal{C}$ be a category with finite products. Then the inclusion*

$$i_n : \mathcal{C} \hookrightarrow \mathcal{C}^{(n)}$$

is cofinal for every n.

Proof. Let $(x_1, \ldots, x_k) \in \mathcal{C}^{(n)}$ where $1 \leq k \leq n$ and consider the category $i_n/(x_1, \ldots, x_k)$. Unwinding the definition of the morphisms in $\mathcal{C}^{(n)}$, it is elementary from the properties of products that the object $x_1 \times \cdots \times x_k \in \mathcal{C}$ is a terminal object of $i_n/(x_1, \ldots, x_k)$, and hence the category is contractible. $\square$

As an immediate corollary, we deduce

Corollary 4.2.4. *Let $\mathcal{C}$ be a category with finite products. Then the restriction map*

$$\operatorname*{holim}_{\mathcal{C}^{(n)}} \Sigma^\infty F^{(n)} \to \operatorname*{holim}_{\mathcal{C}} \Sigma^\infty F$$

is an equivalence for every n. In other words, when $\mathcal{C}$ has finite products, the tower is constant. In particular

$$\operatorname*{holim}_{\mathcal{C}^{(\infty)}} \Sigma^\infty F^{(\infty)} \simeq \operatorname*{holim}_{\mathcal{C}} \Sigma^\infty F$$

A particular example of this phenomenon is the category Δ, which one can easily check has finite products. In this case, a functor $F : \Delta \to \mathcal{S}_*$ is just a cosimplicial space, and the corollary tells us that the tower converges if and only if the linear approximation is an equivalence. That is, if and only if the map

$$\Sigma^\infty \operatorname*{holim}_{\Delta} F \to \operatorname*{holim}_{\Delta} \Sigma^\infty F$$

is an equivalence. Since *every* homotopy limit can be written as a homotopy limit over Δ via cosimplicial replacement, this example shows that the category Δ plays a sort of universal role for the convergence of the tower. We will exploit this idea in Chapter 6 where we examine the convergence properties of the tower in more detail.

As a second example, let us show how to derive the results of [1] cited in the introduction. Thus we assume that we have a functor $F : \mathcal{C} \to \mathcal{S}_*$ which is constant, say $F(x) = X$ for all $x \in \mathcal{C}$. Recall that we have a natural map $\pi : \mathcal{C}^{(n)} \to \mathcal{E}_n^{\mathrm{op}}$. Since F is constant, we may regard $F^{(n)}$ as the composition of π with the functor $S_X : \mathcal{E}_n^{\mathrm{op}} \to \mathcal{S}_*$ given by

$$\underline{k} \mapsto X^{\wedge k}$$

Now, for each n, Lemma 3.3.1 asserts that we have an equivalence

$$\operatorname*{holim}_{\mathcal{C}^{(n)}} \Sigma^\infty F^{(n)} \simeq \operatorname*{holim}_{T(\mathcal{C}^{(n)})} \Sigma^\infty F^{(n)} \circ \tau$$

where $T(\mathcal{C}^{(n)})$ is the twisted arrow category of Section 3.3, and $\tau : T(\mathcal{C}^{(n)}) \to \mathcal{C}^{(n)}$ is the natural projection. In other words, we may pass to the twisted arrow category without loss of generality. Hence

$$\operatorname*{holim}_{\mathcal{C}^{(n)}} \Sigma^\infty F^{(n)} \simeq \operatorname*{holim}_{\mathcal{T}(\mathcal{C}^{(n)})} \Sigma^\infty (S_X \circ \pi \circ \tau)$$

by the previous paragraph.

On the other hand, by Proposition 3.3.2, the map

$$T(\pi) : T(\mathcal{C}^{(n)}) \to T(\mathcal{E}_n^{\mathrm{op}})$$

is a fibration of categories, and we deduce from Corollary 3.2.5 that

$$\operatorname*{holim}_{T(\mathcal{C}^{(n)})} \Sigma^\infty (S_X \circ \pi \circ)\tau \simeq \operatorname*{holim}_{T(\mathcal{E}_n^{\mathrm{op}})} \operatorname*{holim}_{T(\pi)^{-1}(\sigma)} \Sigma^\infty (S_X \circ \pi \circ \tau)$$

where we have let $\sigma : \underline{k} \to \underline{l}$ denote an arbitrary surjection, that is, object of $T(\mathcal{E}_n^{\mathrm{op}})$. But for any $f : (x_1, \ldots, x_l) \to (x'_1, \ldots, x'_k) \in T(\pi)^{-1}(\sigma)$, we have

$$\Sigma^\infty (S_X \circ \pi \circ \tau)(f) = \Sigma^\infty X^{\wedge k}$$

by tracing through the definitions. That is to say, this functor is *also* constant on the subcategory $T(\pi)^{-1}(\sigma)$. Moreover, one easily checks that we have a cofinal inclusion

$$\mathcal{C}^{\times l} \to T(\pi)^{-1}(\sigma)$$

Then it follows from Example 2.2.1 that

$$\begin{aligned} \operatorname*{holim}_{T(\pi)^{-1}(\sigma)} \Sigma^\infty X^{\wedge k} &\simeq \mathrm{Map}(\mathcal{N}(\mathcal{C}^{\times l}), \Sigma^\infty X^{\wedge k}) \\ &\simeq \mathrm{Map}(\Sigma^\infty \mathcal{N}(\mathcal{C})_+^{\times l}, \Sigma^\infty X^{\wedge k}) \\ &\simeq \mathrm{Map}(\Sigma^\infty (\mathcal{C}_+)^{\wedge l}, \Sigma^\infty X^{\wedge k}) \end{aligned}$$

Finally, stringing together the equivalences of the discussion, we find

$$\operatorname*{holim}_{\mathcal{C}^{(n)}} \Sigma^\infty F^{(n)} \simeq \operatorname*{holim}_{T(\mathcal{E}_n^{\mathrm{op}})} \mathrm{Map}(\Sigma^\infty (\mathcal{C}_+)^{\wedge l}, \Sigma^\infty X^{\wedge k})$$

But the right side is now by definition equal to

$$h\,\mathrm{Nat}_{\mathcal{E}_n^{\mathrm{op}}}(\Sigma^\infty(\mathcal{C}_+)^{\wedge k}, \Sigma^\infty X^{\wedge k})$$

which coincides with Theorem 1.0.1 of the Introduction.

4.3 Layers in the Stabilization Tower

Our object now is to analyze the homotopy fiber of the map

$$\operatorname*{holim}_{\mathcal{C}^{(n)}} \Sigma^\infty F^{(n)} \to \operatorname*{holim}_{\mathcal{C}^{(n-1)}} \Sigma^\infty F^{(n-1)}$$

We will present two descriptions of the homotopy fiber in question.

First, observe that we have a diagram of categories

$$\begin{tikzcd} \mathcal{C}^{(n-1)} \arrow[r] \arrow[d] & \mathcal{C}^{(n)} \\ * & \end{tikzcd}$$

where the vertical map is the unique map to the terminal category consisting of a single object and no non-identity morphisms, and the horizontal map is just the inclusion of subcategories. This diagram is given by a functor $Q : \mathcal{K}^{\mathrm{op}} \to \mathcal{C}at$ where $\mathcal{K}^{\mathrm{op}}$ is the *pushout category*. We set

$$\mathcal{C}^{(n)}/\mathcal{C}^{(n-1)} = \nabla Q$$

This construction has the effect of "coning off" the subcategory $\mathcal{C}^{(n-1)}$. In fact, the nerve of this category is nothing other than the homotopy cofiber of the inclusion of nerves $\mathcal{N}(\mathcal{C}^{(n-1)}) \to \mathcal{N}(\mathcal{C}^{(n)})$.

Observe that this category is fibered over $\mathcal{K}$, the pullback category, by construction. Moreover, the functor $\Sigma^\infty F^{(n)}$ may be extended to this category by sending the cone point to the terminal object of $\mathcal{S}p$. Then an application of Corollary 3.2.5 shows

Proposition 4.3.1. *The homotopy fiber of the map*

$$\operatorname*{holim}_{\mathcal{C}^{(n)}} \Sigma^\infty F^{(n)} \to \operatorname*{holim}_{\mathcal{C}^{(n-1)}} \Sigma^\infty F^{(n-1)}$$

is weakly equivalent to

$$\operatorname*{holim}_{\mathcal{C}^{(n)}/\mathcal{C}^{(n-1)}} \Sigma^\infty F^{(n)}$$

One disadvantage of this description is that there is quite a bit of redundant information in the category $\mathcal{C}^{(n)}/\mathcal{C}^{(n-1)}$. This construction leaves us with a much *larger* category, which is nonetheless sometimes useful, as we will see in the next chapter in the case of products.

We can obtain an alternate description of the layers in the stabilization tower by first subdividing the category $\mathcal{C}^{(n)}$ using the twisted arrow construction of Section 3.3. By Lemma 3.3.1, the above map agrees with the map

$$\operatorname*{holim}_{T(\mathcal{C}^{(n)})} \Sigma^\infty F^{(n)} \to \operatorname*{holim}_{T(\mathcal{C}^{(n-1)})} \Sigma^\infty F^{(n-1)}$$

up to a natural weak equivalence, and hence has so does its homotopy fiber.

By construction, the category $\mathcal{C}^{(n)}$ is opfibered over $\mathcal{E}_n$, the category of non-empty finite sets and surjections of cardinality at most n. In particular, if $\mathcal{C} = \{*\}$ is the terminal category, then $\mathcal{C}^{(n)} \cong \mathcal{E}_n$. Hence the category $\mathcal{E}_n$ turns out to be a kind of universal example for the computation of the fiber we are interested in, so we turn first to an analysis of its twisted arrow category.

First, consider the category $T(\mathcal{E}_n)^{\mathrm{op}}$. An object in this category is a surjection $\sigma : p \to q$ where $p, q \leq n$, while a morphism $\sigma \to \tau$ is a commutative square

$$\begin{tikzcd} p \arrow[r] \arrow[d, "\sigma"'] & r \arrow[d, "\tau"] \\ q & s \arrow[l] \end{tikzcd}$$

Consider the three full subcategories defined by

$$\mathcal{A}_0 = \{\sigma : p \to q \mid p = n, q = n-1\}$$
$$\mathcal{A}_1 = \{\sigma : p \to q \mid q = n\}$$
$$\mathcal{A}_2 = \{\sigma : p \to q \mid p \leq n-1\}$$

Notice that if $\sigma \in \mathcal{A}_0$ and $\tau \in \mathcal{A}_1$, then the fact that all maps under consideration are surjections means that we can only have morphisms $\sigma \to \tau$, and not the other way around. The same observation holds for the category $\mathcal{A}_2$. This shows that there is a well defined functor

$$T(\mathcal{E}_n)^{\mathrm{op}} \to \mathcal{K}^{\mathrm{op}}$$

where $\mathcal{K}^{\mathrm{op}}$ is the pushout category, which we depict here in order to fix notation

$$\begin{tikzcd} 0 \arrow[r, "h"] \arrow[d, "k"'] & 1 \\ 2 & \end{tikzcd}$$

The functor in question is defined by sending all the objects of the subcategory $\mathcal{A}_i$ to i.

Lemma 4.3.2. *The functor*

$$T(\mathcal{E}_n)^{\mathrm{op}} \to \mathcal{K}^{\mathrm{op}}$$

is an opfibration.

Proof. The only nontrivial arrows in the base category are h and k. We here exhibit a cocartesian lift over k, the argument for h being similar. Let $\sigma : n \to n-1 \in \mathcal{A}_0$. We claim that the arrow the diagram

$$\begin{tikzcd} n \arrow[r, "\sigma"] \arrow[d, "\sigma"'] & n-1 \arrow[d, "\mathrm{id}"] \\ n-1 & n-1 \arrow[l, "\mathrm{id}"] \end{tikzcd}$$

represents a cocartesian lift for the pair (σ, k). This is because given any other morphism $(\lambda_0, \lambda_1) : \sigma \to \tau$ with $\tau \in \mathcal{A}_2$

$$\begin{tikzcd} n \arrow[r, "\lambda_0"] \arrow[d, "\sigma"'] & p \arrow[d, "\tau"] \\ n-1 & q \arrow[l, "\lambda_1"] \end{tikzcd}$$

we must have $p = q = n-1$ since q is a surjection and $p \leq n-1$. But observe that this diagram factors uniquely as

$$\begin{tikzcd} n \arrow[r, "\sigma"] \arrow[d, "\sigma"'] & n-1 \arrow[r, "\tau^{-1}\lambda_1^{-1}"] \arrow[d, "\mathrm{id}"'] & p = n-1 \arrow[d, "\tau"] \\ n-1 & n-1 \arrow[l, "\mathrm{id}"] & q = n-1 \arrow[l, "\lambda_1"] \end{tikzcd}$$

which shows the arrow $(\sigma, \mathrm{id}_{n-1})$ is cocartesian. $\square$

Corollary 4.3.3. *For any small category $\mathcal{C}$, the category $T(\mathcal{C}^{(n)})$ is fibered over the pullback category $\mathcal{K}$.*

Proof. We have already remarked that $\mathcal{C}^{(n)}$ is opfibered over $\mathcal{E}_n$. Hence by Proposition 3.3.2 and Lemma 4.3.2, the composite

$$T(\mathcal{C}^{(n)}) \to T(\mathcal{E}_n) \to \mathcal{K}$$

is a fibration. $\square$

We denote the fibration constructed in Corollary 4.3.3 by

$$\pi^n_{\mathcal{C}} : T(\mathcal{C}^{(n)}) \to \mathcal{K}$$

It will be convenient to have a more concrete description of the fibers, which we now pursue. First, consider $(\pi^n_{\mathcal{C}})^{-1}(1)$. The objects of this category consist of all arrows $(x_1, \dots, x_n) \to (y_1, \dots, y_n)$ between n-tuples of objects of $\mathcal{C}$. We can describe the situation as follows: the category $\mathcal{C}^{\times n}$ has a natural action of the n-th symmetric group Σ_n. Considering Σ_n as a one object category, this means we have a functor

$$s^n_{\mathcal{C}} : \Sigma_n^{\mathrm{op}} \to \mathcal{C}at$$

sending the unique object to $\mathcal{C}^{\times n}$. (The reason for the opposite appearing here is that this action is merely the restriction of our functor $\mathcal{E}^{\mathrm{op}} \to \mathcal{C}at$ to the set with n elements.) In order to simplify the notation, we set

$$\Sigma_n^{\mathrm{op}}\mathcal{C} = \int s^n_{\mathcal{C}}$$

Unraveling the definitions, we find that

$$(\pi^n_{\mathcal{C}})^{-1}(1) \cong T(\Sigma_n^{\mathrm{op}}\mathcal{C})$$

The subcategory $(\pi^n_{\mathcal{C}})^{-1}(2) \subseteq T(\mathcal{C}^{(n)})$, on the other hand is canonically identified with $T(\mathcal{C}^{(n-1)})$. Finally, the category $(\pi^n_{\mathcal{C}})^{-1}(0)$ is slightly more mysterious. It plays the role in this setup of the *fat diagonal* in [1]. For now, we simply assign it a name

$$D_n\mathcal{C} = (\pi^n_{\mathcal{C}})^{-1}(0)$$

Proposition 4.3.4. *Let $F : \mathcal{C} \to \mathcal{S}_*$ be a $\mathcal{C}$-diagram of pointed, fibrant simplicial sets. The the square*

$$\begin{array}{ccc} \operatorname{holim}_{T(\mathcal{C}^{(n)})} \Sigma^\infty F^{(n)} & \longrightarrow & \operatorname{holim}_{T(\Sigma_n^{\mathrm{op}}\mathcal{C})} \Sigma^\infty F^{(n)} \\ \downarrow & & \downarrow \\ \operatorname{holim}_{T(\mathcal{C}^{(n-1)})} \Sigma^\infty F^{(n-1)} & \longrightarrow & \operatorname{holim}_{D_n\mathcal{C}} \Sigma^\infty F^{(n)} \end{array}$$

is a homotopy pullback.

Proof. As we have shown that the map

$$\pi_{\mathcal{C}}^n : T(\mathcal{C}^{(n)}) \to \mathcal{K}$$

is a fibration of categories, the statement of the proposition is just an application of Corollary 3.2.5 with base category $\mathcal{K}$, the pullback category. $\square$

Consider the subcategory of $T(\mathcal{C}^{(n)})^{\mathrm{op}}$ spanned by both $T(\Sigma_n^{\mathrm{op}}\mathcal{C})^{\mathrm{op}}$ and $(D_n\mathcal{C})^{\mathrm{op}}$. This is exactly the subcategory of all arrows $(x_1, \ldots, x_p) \to (y_1, \ldots, y_q)$ where $p, q \geq n-1$. Moreover, we have seen that this subcategory is opfibered over the category consisting of just a single morphism $0 \xrightarrow{h} 1$. As in the first description of the fiber above, we can extend this data to a functor $G : \mathcal{K}^{\mathrm{op}} \to \mathcal{C}at$, that is, a pushout diagram of categories, which we depict as

$$\begin{array}{ccc} 0 & \xrightarrow{h} & 1 \\ {\scriptstyle k}\downarrow & & \\ 2 & & \end{array} \qquad \stackrel{G}{\Longrightarrow} \qquad \begin{array}{ccc} (D_n\mathcal{C})^{\mathrm{op}} & \longrightarrow & T(\Sigma_n^{\mathrm{op}}\mathcal{C})^{\mathrm{op}} \\ \downarrow & & \\ * & & \end{array}$$

where the vertical morphism in the right hand diagram is just the map to the terminal category consisting of a single object.

We use the notation

$$\mathcal{C}^{\times n}/D_n\mathcal{C} = \left(\int G\right)^{\mathrm{op}}$$

For the Grothendieck construction on this diagram. This may be slightly misleading, as the category above has been subdivided using the twisted arrow construction, which is not reflected in the notation. On the other hand, we have seen that this extra subdivision does not really affect the homotopy type of homotopy limits indexed by the category, and the current notation does make the connection with the results of [1] more clear. In particular, we now have

Corollary 4.3.5. *The homotopy fiber of the map*

$$\operatorname*{holim}_{\mathcal{C}^{(n)}} \Sigma^\infty F^{(n)} \to \operatorname*{holim}_{\mathcal{C}^{(n-1)}} \Sigma^\infty F^{(n-1)}$$

is weakly equivalent to

$$\operatorname*{holim}_{\mathcal{C}^{\times n}/D_n\mathcal{C}} \Sigma^\infty F^{(n)}$$

Proof. By construction, the category $\mathcal{C}^{\times n}/D_n\mathcal{C}$ is fibered over the pullback category $\mathcal{K}$. After restricting the functor $F^{(n)}$ to the subcategory consisting of $D_n\mathcal{C}$ and $T(\Sigma_n^{\mathrm{op}}\mathcal{C})$, we then extend it to be the one-point space on the cone point of $\mathcal{C}^{\times n}/D_n$. An application of Corollary 3.2.5 shows that the homotopy limit over this category computes the fiber of the map in question. □

Chapter 5

The Stabilization Tower for Products

In this chapter we specialize to the case when our indexing category $\mathcal{C}$ is discrete. That is, $\mathcal{C} \cong S$ for some finite set S regarded as a discrete category. We show that our decomposition recovers the classical stable splitting of a product. This special case will play a crucial role in our proof of the convergence for arbitrary homotopy limits.

5.1 Stable Splitting of Products

We recall the following classical result

Lemma 5.1.1. *For pointed spaces X and Y, we have*

$$\Sigma^\infty X \times Y \simeq \Sigma^\infty X \vee \Sigma^\infty Y \vee \Sigma^\infty X \wedge Y$$

We now recast this result in terms of homotopy limits. Consider the set $S = \{0, 1\}$ as an indexing category. Then a functor $F : S \to \mathcal{S}_*$ is just an S-indexed collection of pointed spaces, in the current case, just a pair $X = F(0)$ and $Y = F(1)$. Moreover, we have

$$\operatorname*{holim}_{S} F \simeq X \times Y$$

On the other hand, we can view each of the terms appearing on the right hand side in Lemma 5.1.1 as a smash product over the non-empty subsets $\{0\}, \{1\}, \{0,1\}$ of S. We write $\mathcal{P}_+(S)$ for the set of non-empty subsets of S. Furthermore, in the stable category, the wedge appearing on the right side of the equivalence in Lemma 5.1.1 *is* the categorical product. Hence we can write

$$\prod_{U \in \mathcal{P}_+(S)} \Sigma^\infty \left(\bigwedge_{u \in U} F(u) \right)$$

Lemma 5.1.2. *For a finite set S and functor $F : S \to \mathcal{S}_*$, we have*

$$\Sigma^\infty \left(\prod_{s \in S} F(s) \right) \simeq \prod_{U \in \mathcal{P}_+(S)} \Sigma^\infty \left(\bigwedge_{u \in U} F(u) \right)$$

Proof. Apply induction on the cardinality of S. □

5.2 Convergence for Finite Products

We now consider a finite set S of cardinality k regarded as a discrete category and analyze the category $S^{(n)}$. We observe here that the analysis carried out below carries over without change to the case $n = \infty$.

An object of $S^{(n)}$ is a p-tuple of elements of S which we denote $(x_1, \ldots, x_p)$ where $1 \leq p \leq n$. Since by assumption $|S| = k$, there can be at most k distinct elements among the x_i. By regarding the components of the p-tuple $(x_1, \ldots, x_p)$ as a finite set in their own right, we obtain a subset of S which we refer to as the *characteristic subset* of $(x_1, \ldots, x_p)$. We write

$$\chi(x_1, \ldots, x_p) = \{x_1, \ldots, x_p\} \subseteq S$$

Thus χ has the effect of removing the duplicate among the p-tuple $(x_1, \ldots, x_p)$. In particular, we may have $|\chi(x_1, \ldots, x_p)| < p$. An p-tuple $(x_1, \ldots, x_p)$ for which $|\chi(x_1, \ldots, x_p)| = p$ will be refered to as a *generating object*, and in this case, we must have $p \leq k = |S|$.

As there are no non-identity morphisms in S, a morphism $(x_1, \ldots, x_q) \to (x_1, \ldots, x_p)$ is completely determined by a surjection $\sigma : \underline{p} \to \underline{q}$ and by the property that $x_i = x_{\sigma(i)}$ for all $1 \leq i \leq p$. In particular, the surjectivity of σ implies that every one of the x_i must appear among the collection of elements $\{x_1, \ldots, x_q\}$ so that we must have $\chi(x_1, \ldots, x_q) = \chi(x_1, \ldots, x_p)$. (Informally, a morphism in $S^{(n)}$ can only duplicate elements which already exist in the domain, hence the distinct elements of the domain and codomain coincide.) This shows that χ is constant on the connected components of $S^{(n)}$ and hence can be regarded as a functor

$$\chi : S^{(n)} \to \mathcal{P}_+^{\leq n}(S)$$

where $\mathcal{P}_+^{\leq n}(S)$ is the set of non-empty subsets of S of cardinality at most n, regarded as a discrete category.

Choose a total ordering on the finite set S and let $U \neq \emptyset \subseteq S$. We may write $U = \{x_1, \ldots, x_q\}$ where $x_1 < \cdots < x_q$ in the ordering on S. If $|U| = q \leq n$, then the ordering gives us a well defined object $(x_1, \ldots, x_q) \in S^{(n)}$.

$$\kappa_n : \mathcal{P}_+^{\leq n} \to S^{(n)}$$

and since the x_i are all distinct, $\chi(x_1, \ldots, x_q) = \{x_1, \ldots, x_q\} = U$ so that we have $\chi \circ \kappa_n = \mathrm{id}$.

Now let $(x_1, \ldots, x_p)$ be any other object of $S^{(n)}$ such that $\chi(x_1, \ldots, x_p) = U$. We constuct a surjection $\sigma : \underline{p} \to \underline{q}$ as follows. Set

$$\sigma(i) = j \iff x_i = x_j$$

where $1 \leq i \leq p$ and $1 \leq j \leq q$. This map is clearly unique, which shows that the object

$\kappa_n(U) = (x_1, \ldots, x_q)$ is initial in $\chi^{-1}(U)$. In particular, $\chi^{-1}(U)$ is connected.

Proposition 5.2.1. *Let S be a finite set regarded as a discrete category.*

1. *There is a morphism $(x_1, \ldots, x_q) \to (x'_1, \ldots, x'_p) \in S^{(n)}$ if an only if the domain and codomain have the same characteristic subset.*

2. *The connected components of $S^{(n)}$ are in bijective correspondence with the subsets of S of cardinality at most n.*

3. *Any generating object is initial for its connected component in $S^{(n)}$.*

4. *The functor $\kappa_n : \mathcal{P}_+^{\leq n} \to S^{(n)}$ is cofinal.*

Proof. For 1, we have shown above that if we have a morphism $(x_1, \ldots, x_q) \to (x'_1, \ldots, x'_p)$ then we must have $\chi(x_1, \ldots, x_q) = \chi(x'_1, \ldots, x'_p)$. On the other hand, given two objects $(x_1, \ldots, x_q)$ and $(x'_1, \ldots, x'_p)$ for which $\chi(x_1, \ldots, x_q) = \chi(x'_1, \ldots, x'_p)$, we may construct a surjection $\sigma : \underline{p} \to \underline{q}$ as follows. For each $j \in \underline{q}$, there is some $i \in \underline{p}$ such that $x_j = x'_i$, so put $\sigma(i) = j$. Since $q < p$, there may be elements $l \in \underline{p}$ which are not covered by this definition. But for such l, we have $m \in \underline{q}$ such that $x'_l = x_m$, and we define $\sigma(l) = m$. This map is surjective by construction.

We have already observed that the composition $\chi \circ \kappa_n$ is the identity on $\mathcal{P}_+^{\leq n}(S)$, so that we have at least an injective correspondence by associating to each subset U the connected component of $\kappa_n(U)$. Moreover, for any object $(x_1, \ldots, x_p)$, the object $(\kappa_n \circ \chi)(x_1, \ldots, x_p)$ has the same characteristic object as $(x_1, \ldots, x_p)$ and hence there is a map between them by 1. Hence every object is in a connected component associated to some U.

Statement 3 is clear from the dicussion above, and 4 follows from 3, since the the category $\kappa_n/(x_1, \ldots, x_p)$ will always consist of a single point, namely $\chi(x_1, \ldots, x_p)$. $\square$

Proposition 5.2.1 allows us to completely determine the struction of the Goodwillie tower in the case of products.

Corollary 5.2.2. *For $n \geq k$, where $k = |S|$, the restriction map*

$$\operatorname*{holim}_{S^{(n)}} \Sigma^\infty F^{(n)} \to \operatorname*{holim}_{S^{(k)}} \Sigma^\infty F^{(k)}$$

is an equivalence.

Proof. In view of Proposition 5.2.1, it is easy to check that for any $(x_1, \ldots, x_p)$, the slice category $i_n/(x_1, \ldots, x_p)$ where i_n denotes the inclusion

$$i_n : S^{(k)} \hookrightarrow S^{(n)}$$

has an initial object given by any generating object in the connected component of $(x_1, \ldots, x_p)$. We conclude that i_n is a cofinal functor and the claimed equivalence follows from Proposition 2.2.1. $\square$

Corollary 5.2.3. *We have*

$$\operatorname*{holim}_{S^{(n)}} \Sigma^\infty F^{(n)} \simeq \prod_{U \in \mathcal{P}_+^{\leq n}} \Sigma^\infty \left(\bigwedge_{u \in U} F(u) \right)$$

Proof. From Proposition 5.2.1, the inclusion

$$\kappa_n : \mathcal{P}_+^{\leq n}(S) \to S^{(n)}$$

is cofinal, so the result again follows from Proposition 2.2.1 and the definition of $F^{(n)}$. $\square$

Remark 5.2.1. The description of $\operatorname{holim}_{S^{(n)}} \Sigma^\infty F^{(n)}$ given by the Corollary depends on a choice of ordering on the finite set S and hence is *not* natural with respects to maps of finite sets. In a sense, the advantage of writing what ultimately turns out to be a cartesian product as a homotopy limit over a much larger category is exactly to recover naturality with respect to maps of finite sets.

Theorem 5.2.4. *Let S be a finite set, and $F : S \to \mathcal{S}_*$ a functor. Then the stabilization tower for $\Sigma^\infty \operatorname{holim}_S F$ converges.*

Proof. As we have remarked, the above analysis carries over to the case $n = \infty$. Applying Corollary 5.2.3 and Lemma 5.1.2, we find that

$$\operatorname*{holim}_{S^{(\infty)}} \Sigma^\infty F^{(\infty)} \simeq \prod_{U \in \mathcal{P}_+(S)} \Sigma^\infty \left(\bigwedge_{u \in U} F(u) \right) \simeq \Sigma^\infty \left(\prod_{i \in S} X_i \right) \simeq \Sigma^\infty \operatorname*{holim}_{S} F$$

which is what we wanted to prove. □

5.3 Layers in the Stabilization Tower for Products

For $n > k = |S|$, Corollary 5.2.2 shows that the fiber of the restriction map

$$\operatorname*{holim}_{S^{(n)}} \Sigma^\infty F^{(n)} \to \operatorname*{holim}_{S^{(n-1)}} \Sigma^\infty F^{(n-1)}$$

is contractible. We now compute the fiber for $n \leq k$.

Recall from Proposition 4.3.1 that we may compute the fiber of the map in question as a homotopy limit over the category $S^{(n)}/S^{(n-1)}$ which is by definition the category fibered over $\mathcal{K}$ determined by the diagram

$$\begin{tikzcd} S^{(n-1)} \arrow[r] \arrow[d] & S^{(n)} \\ * & \end{tikzcd}$$

From Proposition 5.2.1, we see that there is a natural transformation of diagrams of categories

$$\begin{tikzcd} \mathcal{P}_+^{\leq n-1}(S) \arrow[r] \arrow[d] & \mathcal{P}_+^{\leq n}(S) \\ * & \end{tikzcd} \quad \overset{\widetilde{\kappa}}{\Longrightarrow} \quad \begin{tikzcd} S^{(n-1)} \arrow[r] \arrow[d] & S^{(n)} \\ * & \end{tikzcd}$$

induced by the inclusions $\kappa_n : \mathcal{P}_+^{\leq n}(S) \to S^{(n)}$ which is cofinal on the fibers. One easily checks that the induced functor

$$\mathcal{P}_+^{\leq n}(S)/\mathcal{P}_+^{\leq n-1}(S) \to S^{(n)}/S^{(n-1)}$$

is again cofinal.

The category $\mathcal{P}_+^{\leq n}(S)/\mathcal{P}_+^{\leq n-1}(S)$ is nearly discrete, with one object comprising its own connected component for each subset $U \subseteq S$ such that $|U| = n$, together with one larger connected component linking together all the subsets $U \neq \emptyset \subseteq S$ such that $|U| < n$. This larger component has a projection which retracts its "leaves", and as $\Sigma^\infty F^{(n)}$ takes the same values at these objects we can regard its restriction to this component as obtained by composition with the retraction. But this retraction is visibly cofinal, as in the following diagram.

$$\begin{tikzcd}
\bullet \arrow[d] & \bullet \arrow[d] & \cdots & \bullet \arrow[d] & & & & & \\
\bullet & \bullet & \cdots & \bullet & \Longrightarrow & \bullet & \bullet & \cdots & \bullet \\
& & * \arrow[ull] \arrow[ul] \arrow[ur] & & & & & * \arrow[ull] \arrow[ul] \arrow[ur] &
\end{tikzcd}$$

On the other hand, the category remaining on the right in the above diagram has an initial object, the cone point. The value of $\Sigma^\infty F^{(n)}$ here is contractible by definition. As a result, the connected component of $\mathcal{P}_+^{\leq n}(S)/\mathcal{P}_+^{\leq n-1}(S)$ containing the cone point contributes nothing to the homotopy limit. As the remaining componenets are discrete, we have proved

Proposition 5.3.1. *Let S be a finite set with $|S| = k$. For $n \leq k$ we have*

$$\operatorname{hofib}\left(\operatorname*{holim}_{S^{(n)}} \Sigma^\infty F^{(n)} \to \operatorname*{holim}_{S^{(n-1)}} \Sigma^\infty F^{(n-1)}\right) \simeq \prod_{\substack{U \subseteq S \\ |U| = n}} \Sigma^\infty \left(\bigwedge_{u \in U} F(u) \right)$$

Remark 5.3.1. This calculation is consistent with the interpretation of the stabilization tower in the Goodwiliie Calculus. We see that in the case of discrete categories, the fibers are exactly the n-homogeneous pieces one would expect.

Chapter 6

Convergence

This chapter contains the main result of this thesis: a theorem describing conditions under which the stabilization tower of a homotopy limit in fact converges to the correct homotopy type. As we pointed out in Section 4.2, the category Δ serves as a universal example for convergence exactly because every homotopy limit can be reduced to one indexed by Δ via cosimplicial replacement. Moreover, as Δ has finite products, convergence can be detected at the linear term. That is, for a cosimplicial space $X : \Delta \to \mathcal{S}_*$, the stabilization tower for X converges if an only if the map

$$\Sigma^\infty \operatorname*{holim}_{\Delta} X \to \operatorname*{holim}_{\Delta} \Sigma^\infty X$$

is an equivalence.

In Section 2.4 we pointed out that for a functor $F : \mathcal{C} \to \mathcal{S}_*$, the cosimplicial replacement of F, which we denoted X_F, is Reedy fibrant, and hence

$$\operatorname*{holim}_{\Delta} X_F \simeq \operatorname{Tot} X_F$$

Intuitively, then, we are reduced to finding conditions which ensure we have an equivalence

$$\Sigma^\infty \operatorname{Tot} X_F \simeq \operatorname{Tot} \Sigma^\infty X_F$$

As the partial totalizations Tot_p can be calculated using cubical diagrams (Proposition 2.5.4), our goal will be to establish bounds on the connectivity of these diagrams and argue using Goodwillie's Generalized Blakers-Massey Theorem (Theorem 2.5.3). A key moment is Proposition 6.3.1, where we are able to transfer the connectivity results on our unstable cubes into the stable world. Passing to the inverse limit, we derive our required equivalence.

6.1 Cubes of Injections

Let Δ be the category of finite ordinals and order preserving maps. The objects are the finite sets of the form $[p] = \{0, 1, \ldots, p\}$ with the natural ordering, and the morphisms are just the (weakly) order preserving maps. In what follows we will almost exclusively be working with objects and morphisms in the category Δ, or a category closely related to Δ, and so will will often omit the brackets when it will not cause confusion. Thus $p \in \Delta$ is shorthand for $[p] \in \Delta$, and $x \in p$ means $0 \leq x \leq p$. We write $\Delta_{\mathcal{M}}$ for the subcategory consisting of just the monomorphisms, and $\Delta_{\mathcal{E}}$ for the subcategory consisting of just the epimorphisms.

We let [2] denote the totally ordered set $\{0, 1, 2\}$ regarded as a category. We draw this category here to fix notation for its objects and morphisms.

[2]

$$\begin{tikzcd} 0 \arrow[rr, "i_2"] \arrow[dr, "i_0"'] & & 2 \\ & 1 \arrow[ur, "i_1"'] & \end{tikzcd}$$

Let $\mu : q \to p \in \Delta_{\mathcal{M}}$ be a monomorphism. We write $\mathcal{P}_\mu$ for the full subcategory of $\mathrm{Hom}_{\mathcal{C}at}([2], \Delta_{\mathcal{M}})$ defined by

$$\mathcal{P}_\mu = \{k : [2] \to \Delta_{\mathcal{M}} \mid k(i_2) = \mu\}$$

Thus an object $k \in \mathcal{P}_\mu$ is a functor $k : [2] \to \Delta_\mathcal{M}$ for which $k(i_1) \circ k(i_0) = k(i_2) = \mu$. In particular, each functor determines a factorization of the given monomoprhism μ, and thus we will occasionally refer to the objects of $\mathcal{P}_\mu$ as *factorizations.* We write $k_0 = k(i_0)$ and $k_1 = k(i_0)$. The functor itself is completely determined by these two morphisms together with the object $k(1) \in \Delta$. It will be useful to write k for both the functor itself, and the object $k(1) \in \Delta$, though if any confusion may arise, we will write $[k] = k(1)$ to emphasize that $k(1)$ is an object of Δ. Thus the statement $k = (k_0, k_1) \in \mathcal{P}_\mu$ corresponds to a diagram

$$\begin{tikzcd} q \arrow[rr, "\mu"] \arrow[dr, "k_0"'] & & p \\ & k \arrow[ur, "k_1"'] & \end{tikzcd}$$

in $\Delta_\mathcal{M}$. For $h, k \in \mathcal{P}_\mu$, a morphism $\alpha : h \to k$ is a commutative diagram

$$\begin{tikzcd} & h \arrow[dr, "h_0"] \arrow[dd, "\alpha"] & \\ q \arrow[ur, "h_0"] \arrow[dr, "k_0"'] & & p \\ & k \arrow[ur, "k_0"'] & \end{tikzcd}$$

Observe that we have a canonical functor $\mathrm{ev}_1 : \mathcal{P}_\mu \to \Delta_\mathcal{M}$ defined by evaluation at the object $1 \in [2]$.

Recall that for a finite set S, we write $\mathcal{P}_S$ for the partially ordered set of subsets of S, regarded as a category. Our notation is motivated by the following observation:

Lemma 6.1.1. *Let $\mu : q \to p \in \Delta_\mathcal{M}$ be a monomorphism. Regard $p \setminus \operatorname{im}\mu \subseteq p$ as a finite set by simply forgetting the ordering. Then we have*

$$\mathcal{P}_\mu \cong \mathcal{P}_{p \setminus \operatorname{im}\mu}$$

Proof. The isomophism is specified by sending an object $k \in \mathcal{P}_\mu$ to the subset $\operatorname{im} k_1 \setminus \operatorname{im}\mu$. The rest of the details are left to the reader. □

Hence given a cosimplicial space $X : \Delta \to \mathcal{S}_*$ and a monomorphism $\mu : q \to p \in \Delta_\mathcal{M}$, the composition

$$\mathcal{P}_\mu \xrightarrow{\mathrm{ev}_1} \Delta \xrightarrow{X} \mathcal{S}_*$$

is a cubical diagram in the sense of Definition 2.5.1. We refer to this as the *cube of injections* generated by μ.

It is tempting at this point to abandon use of the categories $\mathcal{P}_\mu$ and work instead with the finite subsets of $p \setminus \operatorname{im} \mu$. For our purposes, this is slightly inadequate. Observe that a morphism $\alpha : h \to k \in \mathcal{P}_\mu$ can be regarded, by means of evaluation, as a legitimate arrow $\alpha : [h] \to [k] \in \Delta$. When there is a cosimplicial space X under consideration, it is much more convenient to retain this interpretation of the morphisms of $\mathcal{P}_\mu$, as X is functorial with respect to their composition.

On the other hand, the identification of $\mathcal{P}_\mu$ with the subsets of $p \setminus \operatorname{im} \mu$ lets us perform certain set-theoretic constructions on the objects $k \in \mathcal{P}_\mu$. For example, given an element $x \in p \setminus \operatorname{im} \mu$, we will often write $x \in k$ when we strictly mean $\exists i \in k$ s.t. $k_1(i) = x$. Moreover, if $x \in p \setminus \operatorname{im} \mu$ and $x \notin k$, then we obtain a *unique* factorization

$$\begin{tikzcd} q \arrow[rr, "\mu"] \arrow[dr] & & p \\ & k \cup \{x\} \arrow[ur] & \end{tikzcd}$$

In short, it will be convenient to slightly blur the distinction between factorizations of μ and subsets of $p \setminus \operatorname{im} \mu$. We will attempt to be precise when any confusion might arise.

For a monomorphism $\mu : q \to p$, we defined the *adjacency* of μ, denoted $\operatorname{adj} \mu$, to be the object of $\mathcal{P}_\mu$ defined by

$$\operatorname{adj} \mu = \{x \in p \mid x = \mu(y) \pm 1 \text{ for some } y \in \mu\}$$

Thus $\operatorname{adj} \mu$ consists of the elements of p which are directly next to elements in the image of μ. If $x \in \operatorname{adj} \mu$, we will refer to x as an *adjacent element*. Observe that for $p - q > 0$, we have $\operatorname{adj}\mu \neq \emptyset$.

Let $\mu : q \to p \in \Delta_{\mathcal{M}}$ be a monomorphism and let $i \in q-1$. The element i is called *internal to* μ if $\mu(i+1) = \mu(i)+1$. We let $\operatorname{int}\mu$ denote the set of internal elements. Observe that it is possible that $\operatorname{int}\mu = \emptyset$. If $k = (k_0, k_1) \in \mathcal{P}_\mu$ is a factorization, then one easily checks that $k_0(i) \in \operatorname{int} k_1$ for each $i \in \operatorname{int}\mu$. Given any subset $U \subseteq \operatorname{int}\mu$, we set $U_k = \{k_0(i) \mid i \in \operatorname{int}\mu\}$.

Finally, for an epimorphism $\sigma : p \to q \in \Delta_{\mathcal{E}}$, we put

$$V_\sigma = \{i \in p \mid \sigma(i) = \sigma(i+1)\}$$

The motivation for this definition is the following: given a q-chain $x_0 \to \cdots \to x_q$ in some small category $\mathcal{C}$, we obtain by composition with σ a p-chain $x'_0 \to \cdots \to x'_p$. Then for all $i \in V_\sigma$, the map $x'_i \to x'_{i+1}$ is an identity in $\mathcal{C}$. Thus one thinks of the elements of V_σ as those which are forced to be degenerate under precomposition with σ.

6.2 The Cubical Diagram Associated to a Functor

Recall from Section 2.4 that for a functor $F : \mathcal{C} \to \mathcal{S}_*$, we have an associated cosimplicial space X_F, called the *cosimplicial replacement* of F, defined by

$$X_F(p) = \prod_{\vec{x} \in \mathcal{C}_p} F(x_p)$$

As we have remarked above, for any monomorphism $\mu : q \to p \in \Delta_{\mathcal{M}}$, the composition

$$\mathcal{P}_\mu \xrightarrow{\mathrm{ev}_1} \Delta \xrightarrow{X_F} \mathcal{S}_*$$

is a cubical diagram which we will denote $\mathcal{F}_\mu$.

Let $p \in \Delta$. Given any subset $U \subseteq p$ such that $i < p$ for all $i \in U$ we define

$$\mathcal{C}_p|U = \{x_0 \to \cdots \to x_p \in \mathcal{C}_p \mid x_i \to x_{i+1} \text{ is not an identity in } \mathcal{C}\ \forall i \in U\}$$

Then if $U \subseteq \operatorname{int}\mu$ is a collection of internal elements of μ, we may define a cubical diagram $\mathcal{F}_\mu^U : \mathcal{P}_\mu \to \mathcal{S}_*$ by

$$\mathcal{F}_\mu^U(k) = \prod_{\overrightarrow{x} \in \mathcal{C}_k | U_k} F(x_k)$$

Observe in particular that $\mathcal{F}_\mu = \mathcal{F}_\mu^\emptyset$.

Proposition 6.2.1. *Let $F : \mathcal{C} \to \mathcal{S}_*$ be a functor, $\mu : q \to p \in \Delta_\mathcal{M}$ a monomorphism, and $U \subseteq \operatorname{int}\mu$ a collection of internal elements. Then we have*

$$\operatorname{tfib} \mathcal{F}_\mu^U = \Omega^{p-q} \prod_{\overrightarrow{x} \in \mathcal{C}_p | (U_p \cup V_\sigma)} F(x_p)$$

for any retraction $\sigma : p \to q$ of μ.

Proof. We proceed by induction. For $p - q = 0$, the result is trivial. For the inductive step, let $j \in \operatorname{adj}\mu$ be an adjacent element. Without loss of generality we may assume that $j + 1 \in \operatorname{im}\mu$, the proof in the alternative case being similar. Consider the cubical diagram $\mathcal{H}$ defined by

$$\mathcal{H}(k) = \begin{cases} \displaystyle\prod_{\overrightarrow{x} \in \mathcal{C}_k | U_k} F(x_k) & j \notin k \\ \displaystyle\prod_{\substack{\overrightarrow{x} \in \mathcal{C}_k | U_k \\ x_j \to x_{j+1} = \mathrm{id}_j}} F(x_k) & j \in k \end{cases}$$

Observe that for $k \in \mathcal{P}_\mu$ with $j \notin k$, the induced map $\mathcal{H}(k) \to \mathcal{H}(k \cup \{j\})$ is the identity. It follows that $\operatorname{tfib}\mathcal{H} \simeq *$.

Moreover, we have a map of cubical diagrams, that is, a natural transformation $\theta : \mathcal{F}^U \to \mathcal{H}$ whose component at $k \in \mathcal{P}_\mu$ will be denoted θ_k. The components θ_k are given by $\theta_k = \mathrm{id}$ for $j \notin k$ and for $j \in k$, θ_k is the projection

$$\prod_{\vec{x} \in \mathcal{C}_k | U_k} F(k) \to \prod_{\substack{\vec{x} \in \mathcal{C}_k | U_k \\ x_j \to x_{j+1} = \mathrm{id}_j}} F(x_k)$$

In particular, θ_k is always a fibration.

Now, since $\mathcal{H}$ is contractible, $\operatorname{tfib} \mathcal{F}^U$ is equivalent to the total fiber of the cube $\mathcal{L}$ defined by

$$k \mapsto \operatorname{hofib} \theta_k$$

But clearly we have

$$\mathcal{L}(k) = \begin{cases} * & j \notin k \\ \prod_{\vec{x} \in \mathcal{C}_k | (U_k \cup j)} F(x_k) & j \in k \end{cases}$$

Put $q' = q \cup \{j\}$ and $U' = U \cup \{j\}$. Observe that our assumptions on j ensure that $j \in \operatorname{int} \mu'$ where μ' is defined by the factorization diagram

$$\begin{tikzcd} q \arrow[rr, "\mu"] \arrow[dr, "d^j"'] & & p \\ & q \cup \{j\} \arrow[ur, "\mu'"'] & \end{tikzcd}$$

so that this makes sense. Now, one easily sees that the homotopy fiber of $\mathcal{L}$ in the direction of j conincides with the cubical diagram

$$\Omega \circ \mathcal{F}^{U'}_{\mu'} : \mathcal{P}_{\mu'} \to \mathcal{S}_*$$

and the result follows by induction. $\square$

Corollary 6.2.2. *Let $F : \mathcal{C} \to \mathcal{S}_*$ be a $\mathcal{C}$-diagram of pointed, fibrant simplicial sets. Let $\mu : q \to p \in \Delta_{\mathcal{M}}$ be a monomorphism. Then*

$$\operatorname{tfib} \mathcal{F}_\mu \simeq \prod_{\substack{x_0 \to \cdots \to x_p \\ \in \mathcal{C}_p | V_\sigma}} \Omega^{p-q} F(x_p)$$

for any retraction $\sigma : p \to q$ of μ.

Proof. This is just the special case $U = \emptyset$ of Proposition 6.2.1. □

6.3 Stabilization of Cosimplicial Spaces

In this section we use Goodwillie's Generalized Blakers-Massey Theorem, Theorem 2.5.3, to exhibit a condition on a functor $F : \mathcal{C} \to \mathcal{S}_*$ under which we have an equivalence

$$\Sigma^\infty \operatorname{Tot} X_F \simeq \operatorname{Tot} \Sigma^\infty X_F$$

where X_F is the cosimplicial replacement of F. This will be the key ingredient in proving convergence of the stabilization tower for $\operatorname{holim}_{\mathcal{C}} F$.

Definition 6.3.1. A finite category $\mathcal{C}$ is called *strongly finite* if $\dim \mathcal{N}(\mathcal{C}) < \infty$. That is, if there exists some k such that every simplex of dimension higher than k is degenerate. It follows that for each $x \in \mathcal{C}$, we have $\dim \mathcal{N}(\mathcal{C}/x) < \infty$ as well. We write

$$\delta(x) = \dim \mathcal{N}(\mathcal{C}/x)$$

Let $F : \mathcal{C} \to \mathcal{S}_*$ be a $\mathcal{C}$-diagram of pointed, fibrant simplicial sets. For a pointed, fibrant simplicial set Z, we let $c(Z)$ denote the connectivity of Z.

Proposition 6.3.1. *Let $F : \mathcal{C} \to \mathcal{S}_*$ be a $\mathcal{C}$-diagram of pointed, fibrant simplicial sets where $\mathcal{C}$ is strongly finite category with $\dim \mathcal{N}(\mathcal{C}) = n$. Assume that for all $x \in \mathcal{C}$ we have*

$$c(F(x)) \geq \delta(x)$$

Then the map

$$\Sigma^\infty \operatorname{Tot}_p X_F \to \operatorname{Tot}_p \Sigma^\infty X_F$$

is $(\lfloor p/n \rfloor - 1)$*-connected.*

Proof. By Proposition 2.5.4, we may compute $\operatorname{Tot}_n X_F$ as a homotopy limit

$$\operatorname{Tot}_p X_F \simeq \operatorname*{holim}_{\Delta_{\mathcal{M}}/p} X_F \circ \pi_p$$

As we observed in Section 2.5, the composition $X_F \circ \pi_p$ can be regarded as a functor on the category $\mathcal{P}^+_{\underline{p+1}}$, that is, as a cubical diagram missing its initial object. We extend this to a cubical diagram denoted $\mathcal{F}_p : \mathcal{P}_{\underline{p+1}} \to \mathcal{S}_*$ by setting $\mathcal{F}_p(\emptyset) = \operatorname{Tot}_p X_F$. Observe that this cubical diagram is cartesian by construction.

Now, a non empty subset $T \subseteq \underline{p+1}$ corresponds to a monomorphism $\mu_T : q \to p$ for some q. Moreover, the cubical diagram $\partial_{\underline{p+1}\backslash T}\mathcal{F}_p$ is exactly the restriction of $\mathcal{F}_p$ to the subcategory $\mathcal{P}_{\mu_T}$, which we denote $\mathcal{F}_{\mu_T}$. By Corollary 6.2.2, we have

$$\operatorname{tfib} \mathcal{F}_{\mu_T} \simeq \prod_{\substack{x_0 \to \cdots \to x_p \\ \in \mathcal{C}_p | V_{\sigma_T}}} \Omega^{p-q} F(x_p)$$

where $\sigma_T : p \to q \in \Delta_{\mathcal{E}}$ is an arbitrary retraction of μ_T. Now, for a fixed chain $x_0 \to \cdots \to x_p \in \mathcal{C}_p/\sigma$, there are *at least* $p - q$ non-identity arrows $x_i \to x_{i+1}$ by definition. Hence if $p - q > \dim \mathcal{N}(\mathcal{C})$, the above product is indexed by the empty set, and we have $\operatorname{tfib} \mathcal{F}_{\mu_T} \simeq *$ which implies that the cubical diagram $\mathcal{F}_{\mu_T}$ is cartesian. For $p - q \leq n$, we have $c(F(x_p)) \geq \delta(x_p) \geq p - q$ so that $\Omega^{p-q}F(x_p)$ is at least *connected*, that is, 0-connected and hence the cube $\mathcal{F}_{\mu_T}$ is 1-cartesian.

Now, in the notation of Theorem 2.5.3, we set $\kappa(T) = 1$ for all $T \subseteq \underline{p+1}$ such that $|T| \leq n$ and $\kappa(T) = \infty$ otherwise. We conclude that the cubical diagram $\mathcal{F}_p$ is d-cocartesian where d is the minimum of

$$(p+1) - 1 + \sum_{\alpha} \kappa(T_\alpha)$$

over all partitions $\{T_\alpha\}$ of $\underline{p+1}$ by non-empty sets. Taking the partition of $\underline{p+1}$ by singletons, we see that this quantity must be finite. Hence no T_α may appear in the partition such that $|T_\alpha| > n$, since in this case we have $\kappa(T_\alpha) = \infty$. Now, there are $\lfloor \frac{p+1}{n} \rfloor$ subsets of cardinality n, and we clearly acheive a minimum in the above expression by selecting these subsets, along with a single subset containing the remaining elements if $n \nmid (p+1)$. For each such T_α we have $\kappa(T_\alpha) = 1$, and hence we may take

$$d = p + \left\lfloor \frac{p+1}{n} \right\rfloor$$

(Strictly speaking, we may take $d+1$ when $n \nmid (p+1)$, but we will not need this below.)

Following the discussion of [10, Remark 1.19], we observe that the functor Σ^∞ preserves d-cocartesian cubes, since it preserves homotopy colimits. Moreover, using the fact that stably we have hofib $= \Omega$ hocofib, one sees easily by induction that a k-cubical diagram of spectra is d-cocartesian if any only if it is $(d-k+1)$-cartesian. (Roughly, we pick up a shift by the dimension of the cube.)

In the case at hand, $k = p+1$ so that the cube $\Sigma^\infty \mathcal{F}_p$ is

$$p + \left\lfloor \frac{p+1}{n} \right\rfloor - (p+1) + 1 = \left\lfloor \frac{p+1}{n} \right\rfloor$$

cartesian. This says that the map

$$\Sigma^\infty \mathcal{F}_p(\emptyset) = \Sigma^\infty \operatorname{Tot}_p X_F \to \operatorname{Tot}_p \Sigma^\infty X_F$$

is $(\lfloor \frac{p+1}{n} \rfloor - 1)$-connected. $\square$

Corollary 6.3.2. *Let $\mathcal{C}$ be a strongly finite category with $\dim \mathcal{N}(\mathcal{C}) = n$ and $F : \mathcal{C} \to \mathcal{S}_*$ a $\mathcal{C}$-diagram of pointed, fibrant simplicial sets with the property that*

$$c(F(x)) \geq \delta(x)$$

for all $x \in \mathcal{C}$. Then we have an equivalence

$$\Sigma^\infty \operatorname{Tot} X_F \to \operatorname{Tot} \Sigma^\infty X_F$$

Proof. The expression $\lfloor \frac{p+1}{n} \rfloor - 1 \to \infty$ as $p \to \infty$, so we see that for each k we may choose N_k such that for $p \geq N_k$ we have an isomorphism

$$\pi_k(\Sigma^\infty \operatorname{Tot}_p X_F) \xrightarrow{\sim} \pi_k(\operatorname{Tot}_p \Sigma^\infty X_F)$$

Passing to inverse limits, we find that $\pi_k(\Sigma^\infty \operatorname{Tot} X_F) \cong \pi_k(\operatorname{Tot} \Sigma^\infty X_F)$ for all k and the result follows. (In the language of [3], the two Tot towers are *pro-isomorphic*.) $\square$

Remark 6.3.1. As we pointed out in Section 2.4, the cosimplicial space X_F associated to a funtor is always Reedy fibrant. Hence we find that under the conditions of Corollary 6.3.2, we have an equivalence

$$\Sigma^\infty \operatorname*{holim}_{\Delta} X_F \xrightarrow{\sim} \operatorname*{holim}_{\Delta} \Sigma^\infty X_F$$

This formulation will prove useful in the next section.

6.4 Convergence of the Stabilization Tower

In the previous section we saw that if $F : \mathcal{C} \to \mathcal{S}_*$ is a $\mathcal{C}$-diagram of pointed, fibrant simplicial sets with $\mathcal{C}$ strongly finite and such that for all $x \in \mathcal{C}$ we have

$$c(F(x)) \geq \delta(x)$$

then there is an equivalence

$$\Sigma^\infty \operatorname*{holim}_{\Delta} X_F \to \operatorname*{holim}_{\Delta} \Sigma^\infty X_F$$

where X_F is the cosimplicial replacement of F. Recall that the cosimplicial replacement X_F is defined by

$$X_F(p) = \prod_{x_0 \to \cdots \to x_p} F(x_p)$$

where $x_0 \to \cdots \to x_p \in \mathcal{C}_p$ denotes an arbitrary p-chain in $\mathcal{C}$. We observe that there is a "target" functor $T : \mathcal{C}_p \to \mathcal{C}$ sending each p-chain to its last object, and that $F(x_p) = (F \circ T)(x_0 \to \cdots \to x_p)$. If the category $\mathcal{C}$ is strongly finite, then the collection of all p-chains in $\mathcal{C}$ forms a finite set, that is to say, we may regard $\mathcal{C}_p$ as discrete, finite category. Hence, in this case, by Theorem 5.2.4 we may continue the above equivalence:

$$\operatorname*{holim}_{\Delta} \Sigma^\infty X_F \simeq \operatorname*{holim}_{\Delta} \Sigma^\infty \operatorname*{holim}_{\mathcal{C}_p} F \circ T \simeq \operatorname*{holim}_{\Delta} \operatorname*{holim}_{(\mathcal{C}_p)^{(\infty)}} \Sigma^\infty (F \circ T)^{(\infty)}$$

The iterated homotopy limit appearing on the right hand side allows us to apply Corollary 3.2.5 by constructing a new category fibered over Δ. Define a functor $P : \Delta^{\mathrm{op}} \to \mathcal{C}at$ by

$$p \mapsto (\mathcal{C}_p)^{(\infty)}$$

As the objects of $(\mathcal{C}_p)^{(\infty)}$ are n-tuples of p-chains in $\mathcal{C}$, this is functorial over maps $\mu : q \to p \in \Delta$ by forming the composite chain component-wise. Now set

$$\Delta\mathcal{C}^{(\infty)} = \nabla P$$

One should view the category $\Delta\mathcal{C}^{(\infty)}$ as a kind of fattened version of the simplex category of $\mathcal{C}^{(\infty)}$. Unwinding the definitions, we find that an object of $\Delta\mathcal{C}^{(\infty)}$ is a triple $(p, \underline{n}, (\overrightarrow{x}^1, \ldots \overrightarrow{x}^n))$ where $p \in \Delta$, $\underline{n}$ is a finite set with n elements, and each $\overrightarrow{x}^i$ is a p-chain in $\mathcal{C}$. Notice that n and p are determined by the number of components in the tuple and the length of the chains respectively, so we will often omit them from the notation.

There is a functor

$$\widetilde{T} : \Delta\mathcal{C}^{(\infty)} \to \mathcal{C}^{(\infty)}$$

defined by $\widetilde{T}(\overrightarrow{x}^1, \ldots, \overrightarrow{x}^n) = (T(\overrightarrow{x}^1), \ldots, T(\overrightarrow{x}^n)) = (x^1_p, \ldots, x^n_p)$ where we have put $\overrightarrow{x}^i = x^i_0 \to \cdots \to x^i_p$ in agreement with our usual conventions on p-chains. Notice, however, that this necessitates the use of superscripts to index the components of the object $(x^1_p, \ldots, x^n_p) \in \mathcal{C}^{(\infty)}$. We will continue with this convention in the proof of the next lemma.

Lemma 6.4.1. *The functor*

$$\widetilde{T} : \Delta\mathcal{C}^{(\infty)} \to \mathcal{C}^{(\infty)}$$

is cofinal. Moreover, we have $F^{(\infty)} \circ \widetilde{T} = (F \circ T)^{(\infty)}$.

Proof. Let $(x^1, \ldots, x^n) \in \mathcal{C}^{(\infty)}$. Our aim is to show that the category $\widetilde{T}/(x^1, \ldots, x^n)$ is contractible. An object of this category is a k-tuple of q-chains $(\overrightarrow{y}^1, \ldots, \overrightarrow{y}^k) \in \Delta\mathcal{C}^{(\infty)}$ together with a map

$$(y^1_q, \ldots, y^k_q) \to (x^1, \ldots, x^n)$$

in $\mathcal{C}^{(\infty)}$. Recall that such a map is determined by a surjection $\sigma : \underline{n} \to \underline{k}$ and a family of morphisms $\{f_i : y^{\sigma(i)}_q \to x^i\}_{i=1}^n$. This surjection σ determines a map

$$\sigma^* : (\overrightarrow{y}^1, \ldots, \overrightarrow{y}^k) \to (\overrightarrow{y}^{\sigma(1)}, \ldots, \overrightarrow{y}^{\sigma(n)})$$

in $\Delta\mathcal{C}^{(\infty)}$. Now, for each i, the map $f_i : y^{\sigma(i)} \to x^i$ allows us to construct a new chain

$$\overrightarrow{z}^i = y^{\sigma(i)}_0 \to \cdots \to y^{\sigma(i)}_q \xrightarrow{f_i} x^i$$

and the coface map $d^0 : q \to q+1$ induces morphism $(\overrightarrow{y}^{\sigma(1)}, \ldots, \overrightarrow{y}^{\sigma(n)}) \xrightarrow{d^0_*} (\overrightarrow{z}^1, \ldots, \overrightarrow{z}^n)$. Observe that by construction $(\overrightarrow{z}^1, \ldots, \overrightarrow{z}^n) \in \widetilde{T}^{-1}(x^1, \ldots, x^n)$. In fact, the association

$$(\overrightarrow{y}^1, \ldots, \overrightarrow{y}^k) \mapsto (\overrightarrow{z}^1, \ldots, \overrightarrow{z}^n)$$

determines a functor $\widetilde{T}/(x^1, \ldots, x^n) \to \widetilde{T}^{-1}(x^1, \ldots, x^n)$ and the composition $d^0_* \circ \sigma^*$ is a natural transformaion from the identity on $\widetilde{T}/(x^1, \ldots, x^n)$ to its image under this functor. We conclude that the nerves of these two categories are simplicially homotopy equivalent, and thus we are reduced to showing that the category $\widetilde{T}^{-1}(x^1, \ldots, x^n)$ is contractible.

But notice that $(x^1, \ldots, x^n) \in \widetilde{T}^{-1}(x^1, \ldots, x^n)$ by considering it as an n-tuple of 0-chains. For any other object $(\overrightarrow{y}^1, \ldots, \overrightarrow{y}^n) \in \widetilde{T}^{-1}(x^1, \ldots, x^n)$ where each $\overrightarrow{y}^i$ is a p-chain, we have $y^i_p = x^i$ by definition. Now the map $[0] \to [p]$ in Δ which sends $0 \mapsto p$ induces a map

$$(x^1, \ldots, x^n) \to (\overrightarrow{y}^1, \ldots, \overrightarrow{y}^n)$$

in $\widetilde{T}^{-1}(x^1, \ldots, x^n)$. Applying the nerve functor, we find that the collection of such maps forms a contraction of $\mathcal{N}(\widetilde{T}(x^1, \ldots, x^n))$ onto the vertex represented by $(x^1, \ldots, x^n)$.

For the second statement, we have

$$(F^{(\infty)} \circ \widetilde{T})(x^1, \ldots, x^n) = F(x^1) \wedge \cdots \wedge F(x^n) = (F \circ T)^{(\infty)}(x^1, \ldots, x^n)$$

for all objects $(x^1, \ldots, x^n) \in \Delta\mathcal{C}^{(\infty)}$. $\square$

We can now prove out main convergence result.

Theorem 6.4.2. *Let $\mathcal{C}$ be a strongly finite category and let $F : \mathcal{C} \to \mathcal{S}_*$ be a $\mathcal{C}$-diagram of pointed, fibrant simplicial sets such that for all $x \in \mathcal{C}$ we have*

$$c(F(x)) \geq \delta(x)$$

Then the stabilization tower for F converges. That is, we have an equivalence

$$\Sigma^\infty \operatorname*{holim}_{\mathcal{C}} F \xrightarrow{\simeq} \operatorname*{holim}_{\mathcal{C}^{(\infty)}} \Sigma^\infty F^{(\infty)}$$

Proof. In view of the preceding discussion, we have the following chain of equivalences

$$\begin{aligned}
\Sigma^\infty \operatorname*{holim}_{\mathcal{C}} F &\simeq \Sigma^\infty \operatorname*{holim}_{\Delta} X_F \\
&\simeq \operatorname*{holim}_{\Delta} \Sigma^\infty \operatorname*{holim}_{\mathcal{C}_p} F \circ T \\
&\simeq \operatorname*{holim}_{\Delta} \operatorname*{holim}_{(\mathcal{C}_p)^{(\infty)}} \Sigma^\infty (F \circ T)^{(\infty)} \\
&\simeq \operatorname*{holim}_{\Delta\mathcal{C}^{(\infty)}} \Sigma^\infty F^{(\infty)} \circ \widetilde{T} \\
&\simeq \operatorname*{holim}_{\mathcal{C}^{(\infty)}} \Sigma^\infty F^{(\infty)}
\end{aligned}$$

where the last step follows from the cofinality exhibited in Lemma 6.4.1. □

Corollary 6.4.3. *Let $\mathcal{C}$ be a strongly finite category and $F : \mathcal{C} \to \mathcal{S}_*$ a $\mathcal{C}$-diagram of pointed fibrant simplicial sets which is constant at some space X. Suppose that $c(X) > \dim \mathcal{N}(\mathcal{C})$. Then the Goodwillie tower of the functor $\Sigma^\infty \operatorname{Map}(\mathcal{N}(\mathcal{C})_+, X)$ of Theorem 1.0.1 converges.*

Proof. We have already seen in Section 4.2 that the two towers coincide under the hypothesis that F is constant. The convergence of one is then equivalent to the convergence of the other, and our conditions satisfy the requirements of Theorem 6.4.2. □

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