A Note On Left Exact Modalities in Homotopy Type Theory

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1 Introduction

In the theory of modalities, a special role is played by those modalites which preserve identity types. These are the *left-exact* modalities. What makes these modalities particularly interesting, is that they preserve essentially all the structure of type theory, making them a new internal model of type theory. In the connection between type theory and ∞-topos theory, the left-exact modalities correspond to *sub-topoi* [1].

As such, understanding the properties of the left-exact modalities and being able to recognize them becomes of crucial important. In this short note, we prove a recognition theorem for left-exact modalites, as well as an important closure property: we show that the collection of accessible left-exact modalities is closed under the *join* of modalities.

2 A Recognition Criterion

Our recognition criterion will be based on the following characterization of left-exact modalities:

Theorem 2.0.1 ([3] Theorem 3.10)**.** *Let $\bigcirc$ be an accessible modality. The following are equivalent:*

1. *$\bigcirc$ is left-exact*
2. *$\bigcirc$ admits a presentation $P : A \to \mathsf{Type}$ such that for any $a : A$ and any $B : P(a) \to \mathsf{Type}_\bigcirc$ there exists some $Q : \mathsf{Type}_\bigcirc$ such that we have $B(p) \simeq Q$ for all $p : P(a)$.*

Let $P : A \to \mathsf{Type}$ be a dependent family. Recall that any such family gives rise to a modality $\| - \|_P$ called *nullification at P*. We will write $[-]_P$ for the corresponding term former.

Theorem 2.0.2. *Suppose that for every $a : A$ and every $p, q : P(a)$ the identity type $\mathsf{Id}_{P(a)}(p, q)$ is P-connected. That is, we have*

$$\mathsf{is\text{-}contr}\, \|\mathsf{Id}_{P(a)}(p, q)\|_P$$

Then the modality generated by P is left-exact.

Proof. We will use the characterization of Theorem 2.0.1. To this end, we suppose that for each $a : A$ we are given some $B : P(a) \to \mathsf{Type}$. We will take

$$Q := \prod_{q:P(a)} B(q)$$

Hence it remains to construct for each $a : A$ and $p : P(a)$ an equivalence

$$B(p) \simeq \prod_{q:P(a)} B(q)$$

To this end, recall that for $p, q : P(a)$ we have have the function

$$\mathsf{ap} : (p = q) \to (B(p) \simeq B(q))$$

where we have used univalence to write the codomain as a space of equivalences. But since the space $B(p) \simeq B(q)$ is $\bigcirc$-modal, and the space $p = q$ is $\bigcirc$-connected by our hypothesis, this means that the inclusion of the constant functions

$$\mathsf{cst} : B(p) \simeq B(q) \xrightarrow{\simeq} (B(p) \simeq B(q))^{(p=q)}$$

is an equivalence. As the function ap is an element of the codomain of this map, this means that to each $p, q : P(a)$ we can associated a unique equivalence $\phi_{p,q} : B(p) \simeq B(q)$ by chosing an element in the fiber over ap. Crucially, this equivalence is *independent of any chosen path between p and q* in the following sense: by consruction, for *any* path $i : p = q$ and any $b : B(p)$ we have

$$\phi_{p,q}(b) = \mathsf{ap}(B)(i)(b)$$

In particular, notice that

$$\phi_{p,p}(b) = \mathsf{ap}(B)(\mathsf{refl})(b) = b$$

Now, fixing $p : P(a)$ we define maps

$$\alpha : B(p) \to \prod_{q:P(a)} B(q) \qquad \beta : \prod_{q:P(a)} B(q) \to B(p)$$

$$\alpha = \lambda b.\lambda q.\phi_{p,q}(b) \qquad \beta = \lambda\sigma.\sigma(p)$$

And we claim that α and β are inverse to each other. In one direction, for $b : B(p)$ we have

$$\beta(\alpha(p)) = \phi_{p,p}(b) = b$$

with the last equality following from the discussion above.

On the other hand, for a section $\sigma : \prod_{q:B(p)} B(q)$ we have that

$$\alpha(\beta(\sigma)) = \lambda q.\phi_{p,q}(\sigma(p))$$

so that by function extensionality, we are reduced to showing that for all $q : P(a)$ we have

$$\sigma(q) = \phi_{p,q}(\sigma(p))$$

But in this case, the goal is P-modal. Hence we may suppose that we are given an element of any P-connected type, which in this case we take to be $p = q$. Consequently, it will suffice to prove this equality in the presence of an equality $i : p = q$. But then we can perform path induction on this path, and as we have already seen that $\phi_{p,p}$ is the identity function, we are done. □

3 Acyclic Inequalities

Definition 3.0.1. Given families $P : A \to \mathsf{Type}$ and $Q : B \to \mathsf{Type}$, we say that P *is dominated by* Q, and write $P \geq Q$ if we have

$$\prod_{a:A} \mathsf{is\text{-}contr}\, \|P(a)\|_Q$$

That is, if the type $P(a)$ is Q-connected for every a.

Definition 3.0.2. We will say that a type family P is *surjective* if the type $P(a)$ is inhabited for all $a : A$.

Example 3.0.3. Let $P : A \to \mathsf{Type}$ be a type family. Then according to Theorem we see that P is left exact if and only if

$$\Delta P \geq P$$

In general, modalities do not preserve identity types. Nonetheless, we can still describe their action on identity types as follows:

Theorem 3.0.4 ([2])**.** *Let $Q : A \to \mathsf{Type}$ be a type family. Then for any type P and elements $p, q : P$ we have*

$$\|p =_P q\|_Q \simeq [p] =_{\|P\|_{\nabla Q}} [q]$$

$$\|\mathsf{Id}_P(p,q)\|_Q \simeq \mathsf{Id}_{\|P\|_{\nabla Q}}([p],[q])$$

Proposition 3.0.5. *Let $P : A \to \mathsf{Type}$ and $Q : B \to \mathsf{Type}$ be type families. Then the following are equivalent:*

1. *P is surjective and $\Delta P \geq Q$*
2. *$P \geq \nabla Q$*

Proof. ⇒ For any $a : A$ and $p, q : P(a)$, we have from Theorem 3.0.4 that

$$\|p =_{P(a)} q\|_Q \simeq [p] =_{\|P(a)\|_{\nabla Q}} [q] \tag{1}$$

Now, if we have that $\Delta P \geq Q$, then by assumption, the left side is contractible, hence so is the right. This implies that $\|P(a)\|_Q$ is a proposition. (In fact, we must show that $\alpha = \beta$ for arbitrary α and β in $\|P(a)\|_Q$. But as this identity type is ∇Q-modal, we may assume α and β are of the form $[p]$ and $[q]$.)

But then since $P(a)$ is also inhabited by assumption, we have that $\|P(a)\|_Q$ is contractible, as required.

$\Leftarrow$ Let $a : A$ and $p, q : P(a)$. Since $P \geq \nabla Q$, we have that $\|P(a)\|_{\nabla Q}$ is contractible, and hence so is the right side of Equation 1. Then the left side is as well which shows that $\Delta P \geq Q$. Finally, we have

$$P \geq \nabla Q \geq S^0$$

so that P is surjective. □

Corollary 3.0.6. *Suppose $P : A \to \mathsf{Type}$ is a surjective type family and that the modality $\| - \|_P$ is left exact. Then every P-connected type is infinitely connected.*

Proof. Since P is left exact, we have $\Delta P \geq P$. Hence since we have assumed tha P is surjective, Proposition 3.0.5 tells us that we have $P \geq \nabla P$. Hence

$$P \geq \nabla^n P \geq S^{n-1}$$

for all n, which gives the result. □

4 The Join of Left Exact Modalities

We now aim to show that left exact modalities are closed under the join operation. So, let us suppose that we are given two families $P : A \to \mathsf{Type}$ and $Q : B \to \mathsf{Type}$. Then we put

$$(P \star Q) : A \times B \to \mathsf{Type}$$
$$(P \star Q)(a, b) = P(a) \star P(b)$$

Theorem 4.0.1. *Let $P : A \to \mathsf{Type}$ and $Q : B \to \mathsf{Type}$ and suppose that both $\| - \|_P$ and $\| - \|_Q$ are left-exact. Then so is $\| - \|_{P \star Q}$.*

Proof. We are going to check that for every $a : A$ and $b : B$ we have

$$\Delta(P(a) \star Q(b)) \geq P \star Q$$

Observe that this statement is a $(P \star Q)$-modal proposition. Hence we may assume that we have some $x : P(a) \star Q(b)$. We now perform induction on x. Moreover, as the statement is a proposition, there is not path case to check.

So, if $x = p : P(a)$, then we have that $P(a)$ is non-empty. Now, we have $\Delta P(a) \geq P$ since P is left exact. But since $P(a)$ is non-empty, we also have $P(a) \geq \nabla P$. On the other hand, we have $Q(b) \geq Q$ tautologically. Hence we obtain:

$$P(a) \star Q(b) \geq \nabla P \star Q$$

And the result now comes from adjointness in the other direction. The case $x = q : Q(b)$ is symmetric. □

References

[1] Mathieu Anel, Georg Biedermann, Eric Finster, and André Joyal. Higher sheaves and left-exact localizations of ∞-topoi, 2021.

[2] J Daniel Christensen, Morgan Peck Opie, Egbert Rijke, and Luis Nerio Scoccola. Localization in homotopy type theory. *Higher Structures*, 4(1), 2020.

[3] Bas Spitters, Michael Shulman, and Egbert Rijke. Modalities in homotopy type theory. *Logical Methods in Computer Science*, 16, 2020.