Homotopy Type Theory - Exercise Sheet 2

1. Let $X : \mathsf{Type}$ and let $P : X \to \mathsf{Type}$ be a dependent type over X. Show that if $x, y : X$ and $p : x \equiv y$ then there is a function:

$$\mathsf{transport}\, P\, p : P\, x \to P\, y$$

2. Show that the $\mathsf{transport}$ map is actually an equivalence.

3. Let $X : \mathsf{Type}$, $x, y, z : X$ and $p : y \equiv z$. Show that

$$\mathsf{transport}\, (\lambda w.x \equiv w)\, p : x \equiv y \to x \equiv z$$

   is given by composition with the path p.

4. Characterize the equalize of a dependent sum as follows: suppose X and P are as above. For elements (x_0, p_0) and (x_1, p_1) in $\sum_{x:X} Px$ and show that

$$((x_0, p_0) \equiv (x_1, p_1)) \simeq (\sum_{p:x \equiv y} \mathsf{transport}\, P\, p\, x_0 \equiv x_1)$$

5. Again let $X : \mathsf{Type}$ and let $P : X \to \mathsf{Type}$ be a dependent type over X. Given $x, y : X$, some $\alpha : x \equiv y$ and elements $p : P\, x$ and $q : P\, y$, we can define a type of "dependent equalities between p and q over α" which we will write

$$p \equiv q\, [P \downarrow \alpha] : \mathsf{Type}$$

   The definition is by induction on α: when α is reflexivity the elements p and q have the same type and we can simply define

$$p \equiv q\, [P \downarrow \mathsf{refl}] := p \equiv q$$

   (a) Show that

$$p \equiv q\, [P \downarrow \alpha] \simeq q \equiv \mathsf{transport}\, P\, \alpha\, p$$

   (b) Give a second characterization of equalities in a $\sum$ type using $\mathsf{PathOver}$.

6. Suppose we have two types $A, B : Type$ and an equivalence $e : A \simeq B$. Write ua for inverse of the map $A \equiv B \to A \simeq B$ asserted by the univalence axiom. Show that

$$\mathsf{transport}\, (\lambda X.X)\, (\mathsf{ua}\, e) : A \to B$$

   is given by simply applying the equivalence e.