Homotopy Type Theory - Exercise Sheet 1

I'm going to simply write $x \equiv_A y$ for what I wrote as $\mathsf{Id}_A(x, y)$ in lecture. I will often drop the "A" from the notation, writing simply $x \equiv y$ if the type is clear from context.

For the following exercise, you will need to read about the *elimination principle* for identity types which we call J. This is covered in section 1.12.1 in the Homotopy Type Theory book.

1. Show that equality is *symmetric* and *transitive*

2. Prove some elementary groupoid laws. I'll write p^{-1} for the symmetry operation defined in the previous exercise. I'll write $p \cdot q$ for the operator corresponding to transitivity.

   Show (for $p : x \equiv y$ and $q : y \equiv z$, $r : z \equiv w$):

   (a) $p \cdot \mathsf{refl} \equiv p$

   (b) $\mathsf{refl} \cdot p \equiv p$

   (c) $p \cdot p^{-1} \equiv \mathsf{refl}$

   (d) $(p \cdot q) \cdot r \equiv p \cdot (q \cdot r)$

3. Prove that functions $f : X \to Y$ preserve the groupoid structure of types:

   (a) There is a function $\mathsf{ap} : x \equiv y \to f\,x \equiv f\,y$

   (b) ap preserves composition:

   $$\mathsf{ap}\,f\,(p \cdot q) \equiv (\mathsf{ap}\,p) \cdot (\mathsf{ap}\,q)$$

   (c) ap preserves symmetry:

   $$\mathsf{ap}\,f\,p^{-1} \equiv (\mathsf{ap}\,f\,p)^{-1}$$

   (d) Define ap_2 which sends "2-paths" in X to appropriately typed "2-paths" in Y

4. Prove that if a type X is contractible, then so is $x = y$ for any $x\,y : X$

5. Prove that being contractible is a proposition

6. Using functional extensionality, show that if $X : \mathsf{Type}$ and $P : X \to \mathsf{Type}$ satisfies $\prod_{x:X} \mathsf{is\text{-}n\text{-}type}\, P\, x$ for some $n : \mathbb{N}$, then we have

$$\mathsf{is\text{-}n\text{-}type}(\prod_{x:X} P\, x)$$

7. **Meta-exercise**: Exercises 2.1-2.4 in the HoTT Book.