Topos Theory - Solutions to Exercise Sheet 4

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1. We start by defining a functor Φ from $\mathcal{S}et^{\mathcal{C}^{\mathrm{op}}}/F$ to $\mathcal{S}et^{\left(\int F\right)^{\mathrm{op}}}$. On objects, Φ takes a natural transformation $\alpha\colon G \to F$ to the presheaf G_α on $\int F$ defined by on objects by

$$G_\alpha(c,x) \coloneqq \{y \in G(c) \mid \alpha_c(y) = x\} \subseteq G(c).$$

On morphisms G_α is defined as the restriction of G, which is well-defined, because if we have $f\colon c \to d$ in $\mathcal{C}$ with $F(f)(y) = x$ and $z \in G(d)$ such that $\alpha_d(z) = y$, then by naturality, $\alpha_c(G(f)(z)) = F(f)(\alpha_d(z)) = F(f)(y) = x$, as required. And G_α is a functor, because G is.

On morphisms Φ takes a natural transformation $\beta\colon G \to G'$ making a diagram

$$\begin{tikzcd} G \arrow[rr, "\beta"] \arrow[dr, "\alpha"'] & & G' \arrow[dl, "\alpha'"] \\ & F & \end{tikzcd}$$

commute and sends it to the restriction of β. This is well-defined, for if we have $x \in F(c)$ and $y \in G(c)$ with $\alpha_c(y) = x$, then $\alpha'_c(\beta_c(y)) = \alpha_c(y) = x$ by the commutativity of the diagram.

We proceed by showing that Φ is full. Note that any natural transformation $\beta\colon G_\alpha \to G'_{\alpha'}$ can be seen as a natural transformation from G to G', because $y \in G(c)$ if and only if $y \in G_\alpha(c, \alpha_c(y))$, and similarly for G' of course. Moreover, if $y \in G(c)$, then $\alpha'_c(\beta_c(y)) = \alpha_c(y)$, so β is in fact an arrow in $\mathcal{S}et^{\mathcal{C}^{\mathrm{op}}}/F$.

Further, Φ is faithful, for if we have β and γ as above such that $\Phi(\beta) = \Phi(\gamma)$, then for every $y \in G(c)$ we have $\beta_c(y) = \Phi(\beta)_{(c,\alpha_c(y))}(y) = \Phi(\gamma)_{(c,\alpha_c(y))}(y) = \gamma_c(y)$, proving that $\beta = \gamma$.

Finally, we show that Φ is (split) essentially surjective on objects. Suppose that $\mathcal{G}$ is a presheaf on $\int F$. We construct a presheaf G on $\mathcal{C}$ over F such that $\Phi(G) \cong \mathcal{G}$. On objects, we define $G(c) \coloneqq \coprod_{x \in F(c)} \mathcal{G}(c,x)$, where $\coprod$ denotes the disjoint union of sets. For an arrow $f\colon c \to d$ in $\mathcal{C}$, we define $G(f)\colon G(d) \to G(c)$ by $(y,z) \mapsto (F(f)(y), \mathcal{G}(f)(z))$, which is functorial, because F and $\mathcal{G}$ are functors.

We equip G with a natural transformation π which is simply given by the first projection, i.e. $\pi_c(x,y) = x$ for all $x \in F(c)$ and $y \in \mathcal{G}(c,x)$. This makes G into an element of $\mathcal{S}et^{\mathcal{C}^{\mathrm{op}}}/F$.

It remains to prove that $\Phi(G) \cong \mathcal{G}$, but this follows readily, as we calculate that for every $(c,x) \in \int F$, we have

$$(\Phi(G))(c,x) = \{(x',y) \in G(c) \mid x' = x\} = \{(x,y) \mid y \in \mathcal{G}(c,x)\} \cong \mathcal{G}(c,x).$$

We conclude that Φ is an equivalence, as desired.

2. We start by showing that F is a cocone for the diagram $\int F \xrightarrow{\pi_F} \mathcal{C} \xrightarrow{\mathbf{y}} \mathcal{S}et^{\mathcal{C}^{\mathrm{op}}}$. We need to define for each $(c,x) \in \int F$ a natural transformation $\phi_x \colon \mathbf{y}(c) \to F$ such that for every $f \colon (c,x) \to (d,y)$ in $\int F$, we have $\phi_y \circ \mathbf{y}(f) = \phi_x$. We define $(\phi_x)_d(g) \coloneqq F(g)(x)$ for an arbitrary object d of $\mathcal{C}$. Note that ϕ_x is a natural transformation by functoriality of F. Moreover, $\phi_y \circ \mathbf{y}(f) = \phi_x$ holds when $F(f)(y) = x$, again by functoriality of F. Thus, F is a cocone for the diagram.

   To see that F is the initial cocone, suppose that (G, ψ) is another cocone, i.e. for every $(c,x) \in \int F$, we have a natural transformation $\psi_x \colon \mathbf{y}(c) \to G$ such that for every $f \colon (c,x) \to (d,y)$ in $\int F$, we have $\psi_y \circ \mathbf{y}(f) = \psi_x$. We show that there is a unique natural transformation $\Psi \colon F \to G$ such that $\Psi \circ \phi_x = \psi_x$ for every $x \in F(c)$.

   We define $\Psi_c \colon F(c) \to G(c)$ by $x \mapsto \psi_x(\mathrm{id}_C)$. This is a natural transformation, because in the diagram

$$\begin{tikzcd} F(d) \arrow[r, "\Psi_d"] \arrow[d, "F(f)"'] & G(d) \arrow[d, "G(f)"] \\ F(c) \arrow[r, "\Psi_c"'] & G(c) \end{tikzcd}$$

   the clockwise composition yields $G(f)(\psi_y(\mathrm{id}_d)) = \psi_y(f)$ by naturality of ψ_y, while the anticlockwise composition reads $\psi_{F(f)(y)}(\mathrm{id}_c) = \psi_y(f)$ because $\psi_y \circ \mathbf{y}(f) = \psi_x$.

   Next, we verify that $\Psi \circ \phi_x = \psi_x$ for every $(c,x) \in \int F$. If $f \colon c' \to c$ is an arrow in $\mathcal{C}$, then

$$\begin{aligned} (\Psi \circ \phi_x)_C(f) &= \Psi_C(F(f)(x)) && \text{(by definition of } \phi_x\text{)} \\ &= \psi_{F(f)(x)}(\mathrm{id}_C) && \text{(by definition of } \Psi\text{)} \\ &= \psi_x(f) && \text{(since } \psi_x \circ \mathbf{y}(f) = \psi_{F(f)(x)}\text{)}, \end{aligned}$$

   as claimed.

   Finally, we claim that Ψ is the unique such natural transformation. For suppose that Θ is another such natural transformation, then for every $x \in F(c)$, we have

$$\begin{aligned} \Theta_C(x) &= \Theta_C(F(\mathrm{id}_c)(x)) && \\ &= \Theta_C((\phi_x)_c(\mathrm{id}_C)) && \text{(by definition of } \phi_x\text{)} \\ &= \psi_x(\mathrm{id}_C) && \text{(since } \Theta \circ \phi_x = \psi_x\text{)} \\ &= \Psi_c(x) && \text{(by definition of } \Psi\text{)}, \end{aligned}$$

   showing that $\Theta = \Psi$. Thus Ψ is unique and F is colimit of the diagram $\int F \xrightarrow{\pi_F} \mathcal{C} \xrightarrow{\mathbf{y}} \mathcal{S}et^{\mathcal{C}^{\mathrm{op}}}$.

3. We describe how $\hat{G}$ acts on a natural transformation $\alpha \colon F \to F'$. By the universal property of the colimit, it suffices to find maps $G(c) \to \hat{G}(F') = \mathrm{colim}(G \circ \pi_{F'})$ for every object c of $\mathcal{C}$. But for such c and $x \in G(c)$ we have $\alpha_c(x) \in F'(c)$ determining an element of $\mathrm{colim}(G \circ \pi_{F'})$ through the colimit inclusions. Moreover, this assignment is clearly functorial. Thus, $\hat{G}$ is indeed a functor.

   A right adjoint to $\hat{G}$ is given by the following data: for every object e of $\mathcal{E}$, a presheaf R_e on $\mathcal{C}$ together with a map $\varepsilon_e \colon \hat{G}(R_e) \to e$ in $\mathcal{E}$ such that for every map $h \colon \hat{G}(F) \to e$ in $\mathcal{E}$, there exists a unique natural transformation $\alpha \colon F \to R_e$ making the diagram

$$\begin{tikzcd} \hat{G}(F) \arrow[d, dashed, "\hat{G}(\alpha)"'] \arrow[dr, "h"] & \\ \hat{G}(R_e) \arrow[r, "\varepsilon_e"'] & e \end{tikzcd} \tag{\dagger}$$

commute. We define $R_e\colon \mathcal{C}^{\mathrm{op}} \to \mathcal{S}et$ on objects by $R_e(c) \coloneqq \{f\colon d \to c \mid G(d) = e\}$. We define $R_e(f)$ for a morphism f to be postcomposition with f in $\mathcal{C}$, which is obviously functorial.

We define $\varepsilon_e\colon \operatorname{colim}(G \circ \pi_{R_e}) \to e$ to be the map induced by taking $G(f)$ at $f\colon d \to c$ with $G(d) = e$.

Finally, if $h\colon \hat{G}(F) \to e$, then h is given by a collection of maps $h_x\colon G(c) \to e$ indexed by $(c, x) \in \int F$ such that

$$h_y \circ G(f) = h_{F(f)(y)} \tag{‡}$$

for $y \in \operatorname{cod}(f)$. Hence, we can define $\alpha\colon F \to R_e$ as the natural transformation given by $\alpha_c(x) \coloneqq \{g\colon d \to c \mid G(g) = h_x\}$. This is indeed a natural transformation because in the diagram

$$\begin{tikzcd} F(d) \arrow[r, "\alpha_d"] \arrow[d, "F(f)"'] & R_e(d) \arrow[d, "f \circ -"] \\ F(c) \arrow[r, "\alpha_c"'] & R_e(c) \end{tikzcd}$$

the clockwise composition reads $\{(f \circ f')\colon \bullet \to c \mid G(f') = h_y\}$, while the anticlockwise composition yields $\{g\colon \bullet \to c \mid G(g) = h_{F(f)(y)}\}$, but these are equal by (‡). It follows from the definitions of α and ε_e that α is the unique natural transformation making (†) commute.