Topos Theory - Solutions to Exercise Sheet 2

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1. For arbitrary elements a, b, c of a distributive lattice, we have

$$\begin{aligned}
(a \vee b) \wedge (a \vee c) &= ((a \vee b) \wedge a) \vee ((a \vee b) \wedge c) && \text{(by distributivity)} \\
&= a \vee ((a \vee b) \wedge c) && \text{(since } a \leq a \vee b\text{)} \\
&= a \vee (a \wedge c) \vee (b \wedge c) && \text{(by distributivity)} \\
&= a \vee (b \wedge c) && \text{(since } a \wedge c \leq a\text{),}
\end{aligned}$$

which shows that the dual distributive law also holds.

2. Recall that $u \Rightarrow v$ is defined such that for every element c we have $c \wedge u \leq v \iff c \leq u \Rightarrow v$. In particular, $c \wedge x = 0 \iff c \leq \neg x$. Since $a \wedge x = 0$, we immediately get that $a \leq \neg x$. For the other inequality, notice that it holds if and only if $c \leq \neg x$ implies $c \leq a$ for every element c. So let c be an arbitrary element with $c \leq \neg x$. Then $c \wedge x = 0$ holds by definition of the Heyting implication. We show that $c \leq a$ holds by proving that $a \vee c = a$. Note:

$$\begin{aligned}
a \vee c &= a \vee (c \wedge 1) \\
&= a \vee (c \wedge (x \vee a)) \\
&= a \vee ((c \wedge x) \vee (c \wedge a)) && \text{(by distributivity)} \\
&= a \vee (0 \vee (c \wedge a)) \\
&= a \vee (c \wedge a) \\
&= (a \vee c) \wedge (a \vee a) && \text{(by dual distributivity)} \\
&= (a \vee c) \wedge a \\
&= a,
\end{aligned}$$

as we wished to show.

3. (a) Let $f\colon X \rightarrowtail A$ and $g\colon X \rightarrowtail B$ be monomorphisms and let $u, v\colon Y \to X$ be arbitrary parallel arrows satisfying $(f, g) \circ u = (f, g) \circ v$. We must show that $u = v$. But $(f, g) \circ u = (f, g) \circ v$ implies $f \circ u = \pi_1 \circ (f, g) \circ u = \pi_1 \circ (f, g) \circ v = f \circ v$, which yields $u = v$, as f is mono.

   NB: Note that the above show that it suffices for just one of f and g to be a mono.

   (b) We start by labelling the arrows.

$$\begin{tikzcd}
A \arrow[r, "a"] \arrow[d, "b"'] & B \arrow[d, "c"] \\
C \arrow[r, "d"'] & D
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
X \arrow[r, "x"] \arrow[d, "y"'] & Y \arrow[d, "z"] \\
Z \arrow[r, "w"'] & W
\end{tikzcd}$$

Suppose that we have arrows $f\colon E \to B \times Y$ and $g\colon E \to C \times Z$ such that the equality $(c, z) \circ f = (d, w) \circ g$ holds. We must show that there is a unique dashed arrow making the diagram

$$\begin{tikzcd}
E \arrow[drr, bend left, "f"] \arrow[dr, dashed] \arrow[ddr, bend right, "g"'] & & \\
& A \times X \arrow[r, "{(a,x)}"] \arrow[d, "{(b,y)}"] & B \times Y \arrow[d, "{(c,z)}"] \\
& C \times Z \arrow[r, "{(d,w)}"] & D \times W
\end{tikzcd}$$

commute. Since $(c, z) \circ f = (d, w) \circ g$, we have

$$(c \circ \pi_1 \circ f, z \circ \pi_2 \circ f) = (d \circ \pi_1 \circ g, w \circ \pi_2 \circ g),$$

and therefore,

$$c \circ \pi_1 \circ f = d \circ \pi_1 \circ g \quad \text{and} \quad z \circ \pi_2 \circ f = w \circ \pi_2 \circ g.$$

Hence, we obtain unique arrows k and l making the diagrams

$$\begin{tikzcd}
E \arrow[drr, bend left, "\pi_1 \circ f"] \arrow[dr, dashed, "k"] \arrow[ddr, bend right, "\pi_1 \circ g"'] & & \\
& A \arrow[r, "a"] \arrow[d, "b"'] & B \arrow[d, "c"] \\
& C \arrow[r, "d"'] & D
\end{tikzcd}
\qquad
\begin{tikzcd}
E \arrow[drr, bend left, "\pi_2 \circ f"] \arrow[dr, dashed, "l"] \arrow[ddr, bend right, "\pi_2 \circ g"'] & & \\
& X \arrow[r, "x"] \arrow[d, "y"'] & Y \arrow[d, "z"] \\
& Z \arrow[r, "w"'] & W
\end{tikzcd}$$

commute. Finally, $(a, x) \circ (k, l) = (a \circ k, x \circ l) = (\pi_1 \circ f, \pi_2 \circ f) = f$ and similarly, we get $(b, y) \circ (k, l) = g$. Moreover, (k, l) is the unique such map, because suppose that $h = (h_1, h_2)\colon E \to A \times X$ is another such map, then $\pi_1 \circ f = a \circ h_1$ and $\pi_1 \circ g = b \circ h_1$, so that $h_1 = k$ by the uniqueness of k. Similarly, one proves that $h_2 = l$, and hence that $h = (k, l)$, finishing the proof.

(c) We define $\wedge\colon \Omega \times \Omega \to \Omega$ to be the classifying map of $(\text{true}, \text{true})\colon 1 \rightarrowtail \Omega \times \Omega$, which is a monomorphism by part (a). Now let $U \rightarrowtail 1$ and $V \rightarrowtail 1$ be arbitrary subterminals. By part (b), the square

$$\begin{tikzcd}
U \times V \arrow[r] \arrow[d] & 1 \times 1 \cong 1 \arrow[d, "\text{true}"] \\
1 \times 1 \cong 1 \arrow[r, "{(\chi_U, \chi_V)}"'] & \Omega
\end{tikzcd}$$

is a pullback. Hence, by pullback pasting the outer rectangle in

$$\begin{tikzcd}
U \times V \arrow[r] \arrow[d] & 1 \arrow[r, equal] \arrow[d, "{(\text{true},\text{true})}"] & 1 \arrow[d, "\text{true}"] \\
1 \arrow[r, "{(\chi_U, \chi_V)}"] & \Omega \times \Omega \arrow[r, "\wedge"'] & \Omega
\end{tikzcd}$$

is a pullback too, which proves that $\wedge \circ (\chi_U, \chi_V)$ classifies $U \times V$ as a subobject of 1.

4. We sketch the constructions, but omit the verifications. We define the morphism $\vee\colon \Omega\times\Omega \to \Omega$ to be the classifying map of the join of the subobjects $\Omega \xrightarrow{(\mathrm{id}_\Omega,\mathrm{true}_\Omega)} \Omega\times\Omega$ and $\Omega \xrightarrow{(\mathrm{true}_\Omega,\mathrm{id}_\Omega)} \Omega\times\Omega$. We define the morphism $\Rightarrow\colon \Omega\times\Omega \to \Omega$ to be the classifying map of the equalizer of $\Omega\times\Omega \overset{\vee}{\underset{\pi_1}{\rightrightarrows}} \Omega$.

5. (a) We start by defining a functor $\chi_-\colon \mathrm{Mono}(\mathcal{E}) \to \mathcal{E}/\Omega$ which sends an object $m\colon A \rightarrowtail B$ of $\mathrm{Mono}(\mathcal{E})$ to its classifying map $\chi_m\colon B \to \Omega$. On morphisms, we send a pullback square

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, tail, "m"'] & X \arrow[d, tail, "n"] \\ B \arrow[r, "g"'] & Y \end{tikzcd}$$

to

$$\begin{tikzcd} B \arrow[rr, "g"] \arrow[dr, "\chi_m"'] & & Y \arrow[dl, "\chi_n"] \\ & \Omega & \end{tikzcd}$$

where we need to check that the triangle commutes, i.e. that $\chi_n \circ g = \chi_m$. But this follows from the uniqueness of classifying maps and the fact that the outer rectangle in the diagram

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, tail, "m"'] & X \arrow[r] \arrow[d, tail, "n"] & 1 \arrow[d, "\mathrm{true}"] \\ B \arrow[r, "g"'] & Y \arrow[r, "\chi_n"'] & \Omega \end{tikzcd}$$

is a pullback, which is true because the left and right squares are pullbacks.

In the other direction, we define a functor $\widetilde{(-)}\colon \mathcal{E}/\Omega \to \mathrm{Mono}(\mathcal{E})$. Given an object $\varphi\colon X \to \Omega$ of $\mathcal{E}/\Omega$, we send it to the mono $\widetilde{\varphi}\colon \widetilde{X} \to X$ given by pulling φ back along $\mathrm{true}\colon 1 \to \Omega$. Given a morphism

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "\varphi"'] & & Y \arrow[dl, "\psi"] \\ & \Omega & \end{tikzcd}$$

in the slice category $\mathcal{E}/\Omega$, we send it to the pullback square

$$\begin{tikzcd} \widetilde{X} \arrow[r, "\bar{f}"] \arrow[d, "\widetilde{\varphi}"'] & \widetilde{Y} \arrow[d, "\widetilde{\psi}"] \\ X \arrow[r, "\varphi"'] & Y \end{tikzcd} \tag{†}$$

where $\bar{f}$ is the unique dashed map making the diagram

$$
\begin{tikzcd}
\widetilde{X} \arrow[dr, dashed, "\bar{f}"] \arrow[drr, bend left] \arrow[ddr, bend right, "f\circ\widetilde{\varphi}"'] & & \\
& \widetilde{Y} \arrow[r] \arrow[d, "\widetilde{\psi}"] & 1 \arrow[d, "\mathrm{true}"] \\
& Y \arrow[r, "\psi"] & \Omega
\end{tikzcd}
$$

commute. Notice that the square (†) is indeed a pullback, because the outer rectangle and right hand square in the diagram

$$
\begin{tikzcd}
\widetilde{X} \arrow[r, "\bar{f}"] \arrow[d, "\widetilde{\varphi}"] & \widetilde{Y} \arrow[r] \arrow[d, "\widetilde{\psi}"] & 1 \arrow[d, "\mathrm{true}"] \\
X \arrow[r, "\varphi"'] & Y \arrow[r, "\psi"] & \Omega
\end{tikzcd}
$$

are both pullbacks.

We omit the verification that these functors constitute an equivalence.

(b) Topoi have finite colimits, so in particular $\mathcal{E}/\Omega$ has finite colimits. But $\mathrm{Mono}(\mathcal{E})$ and $\mathcal{E}/\Omega$ are equivalent categories, so $\mathrm{Mono}(\mathcal{E})$ has finite colimits too.

(c) Once again we only provide a sketch of the proof. Note that the diagram

$$
\begin{tikzcd}
B \arrow[d, tail, "g"] & A \arrow[l] \arrow[r] \arrow[d, tail, "f"] & C \arrow[d, tail, "h"] \\
F & E \arrow[l] \arrow[r] & G
\end{tikzcd}
$$

consists of two pullback squares, so we can consider its pushout in $\mathrm{Mono}(\mathcal{E})$, which we denote by $\sigma\colon S \rightarrowtail T$. As part of the pushout property, we get pullback squares

$$
\begin{tikzcd}
B \arrow[r, "x"] \arrow[d, tail, "g"'] & S \arrow[d, tail, "\sigma"] \\
F \arrow[r] & T
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
C \arrow[r, "y"] \arrow[d, tail, "h"'] & S \arrow[d, tail, "\sigma"] \\
G \arrow[r] & T
\end{tikzcd}
\tag{$\ddagger$}
$$

Now given $u\colon X \to D$ and $v\colon X \to G$ making

$$
\begin{tikzcd}
X \arrow[drr, bend left, "u"] \arrow[ddr, bend right, "u"'] & & \\
& C \arrow[r] \arrow[d, tail, "h"] & D \arrow[d, tail, "k"] \\
& G \arrow[r] & H
\end{tikzcd}
\tag{$\star$}
$$

commute, we describe how to construct a map $X \to C$ which fits the diagram. From the fact that the top square is a pushout, we get the dashed map z making

$$\begin{tikzcd}
A \arrow[r] \arrow[d] & B \arrow[d] \arrow[rdd, bend left, "x"] & \\
C \arrow[r] \arrow[rrd, bend right, "y"'] & D \arrow[rd, dashed, "z"] & \\
& & S
\end{tikzcd}$$

commute, where the maps x and y are as in (‡). Finally, we use the right hand pullback square in (‡) to get a map from X to C as the unique dashed map making the diagram

$$\begin{tikzcd}
X \arrow[rd, dashed] \arrow[rrd, bend left, "z \circ u"] \arrow[rdd, bend right, "v"'] & & \\
& C \arrow[r, "y"] \arrow[d, tail, "h"] & S \arrow[d, tail, "\sigma"] \\
& G \arrow[r] & T
\end{tikzcd}$$

commute. We omit the verification that the dashed map makes the diagram ($\star$) commute and that it is the unique map to do so.