Topos Theory - Solutions to Exercise Sheet 2

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1. For arbitrary elements a, b, c of a distributive lattice, we have

$(a \lor b) \land (a \lor c) = ((a \lor b) \land a) \lor ((a \lor b) \land c)$	(by distributivity)
$= a \vee ((a \vee b) \wedge c)$	(since $a \leq a \lor b$)
$= a \vee (a \wedge c) \vee (b \wedge c)$	(by distributivity)
$= a \vee (b \wedge c)$	(since $a \wedge c \leq a$),

which shows that the dual distributive law also holds.

2. Recall that $u \Rightarrow v$ is defined such that for every element c we have $c \land u \leq v \iff c \leq u \Rightarrow v$. In particular, $c \land x = 0 \iff c \leq \neg x$. Since $a \land x = 0$, we immediately get that $a \leq \neg x$. For the other inequality, notice that it holds if and only if $c \leq \neg x$ implies $c \leq a$ for every element c. So let c be an arbitrary element with $c \leq \neg x$. Then $c \land x = 0$ holds by definition of the Heyting implication. We show that $c \leq a$ holds by proving that $a \lor c = a$. Note:

$$a \lor c = a \lor (c \land 1)$$

= $a \lor (c \land (x \lor a))$
= $a \lor ((c \land x) \lor (c \land a))$ (by distributivity)
= $a \lor (0 \lor (c \land a))$
= $a \lor (c \land a)$
= $(a \lor c) \land (a \lor a)$ (by dual distributivity)
= $(a \lor c) \land a$
= a ,

as we wished to show.

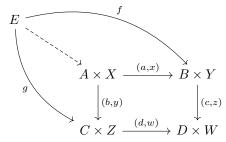
3. (a) Let $f: X \to A$ and $g: X \to B$ be monomorphisms and let $u, v: Y \to X$ be arbitrary parallel arrows satisfying $(f,g) \circ u = (f,g) \circ v$. We must show that u = v. But $(f,g) \circ u = (f,g) \circ v$ implies $f \circ u = \pi_1 \circ (f,g) \circ u = \pi_1 \circ (f,g) \circ v = f \circ v$, which yields u = v, as f is mono.

NB: Note that the above show that it suffices for just one of f and g to be a mono.

(b) We start by labelling the arrows.

$$\begin{array}{cccc} A & \stackrel{a}{\longrightarrow} & B & & X & \stackrel{x}{\longrightarrow} & Y \\ \downarrow^{b} & \downarrow^{c} & & y \downarrow & \downarrow^{z} \\ C & \stackrel{d}{\longrightarrow} & D & & Z & \stackrel{w}{\longrightarrow} & W \end{array}$$

Suppose that we have arrows $f: E \to B \times Y$ and $g: E \to C \times Z$ such that the equality $(c, z) \circ f = (d, w) \circ g$ holds. We must show that there is a unique dashed arrow making the diagram



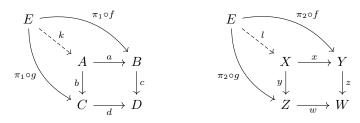
commute. Since $(c, z) \circ f = (d, w) \circ g$, we have

$$(c \circ \pi_1 \circ f, z \circ \pi_2 \circ f) = (d \circ \pi_1 \circ g, w \circ \pi_2 \circ g),$$

and therefore,

 $c \circ \pi_1 \circ f = d \circ \pi_1 \circ g$ and $z \circ \pi_2 \circ f = w \circ \pi_2 \circ g$.

Hence, we obtain unique arrows k and l making the diagrams



commute. Finally, $(a, x) \circ (k, l) = (a \circ k, x \circ l) = (\pi_1 \circ f, \pi_2 \circ f) = f$ and similarly, we get $(b, y) \circ (k, l) = g$. Moreover, (k, l) is the unique such map, because suppose that $h = (h_1, h_2) \colon E \to A \times X$ is another such map, then $\pi_1 \circ f = a \circ h_1$ and $\pi_1 \circ g = b \circ h_1$, so that $h_1 = k$ by the uniqueness of k. Similarly, one proves that $h_2 = l$, and hence that h = (k, l), finishing the proof.

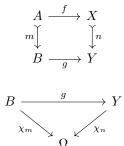
(c) We define $\wedge: \Omega \times \Omega \to \Omega$ to be the classifying map of (true, true): $1 \to \Omega \times \Omega$, which is a monomorphism by part (a). Now let $U \to 1$ and $V \to 1$ be arbitrary subterminals. By part (b), the square

is a pullback. Hence, by pullback pasting the outer rectangle in

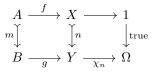
$$\begin{array}{c} U \times V \longrightarrow 1 = 1 \\ \downarrow & \downarrow^{(\text{true,true})} & \downarrow^{\text{true}} \\ 1 \xrightarrow{(\chi_U, \chi_V)} \Omega \times \Omega \xrightarrow{\wedge} \Omega \end{array}$$

is a pullback too, which proves that $\wedge \circ (\chi_U, \chi_V)$ classifies $U \times V$ as a subobject of 1.

- 4. We sketch the constructions, but omit the verifications. We define the morphism $\forall : \Omega \times \Omega \rightarrow \Omega$ Ω to be the classifying map of the join of the subobjects $\Omega \xrightarrow{(id_{\Omega}, true_{\Omega})} \Omega \times \Omega$ and $\Omega \xrightarrow{(true_{\Omega}, id_{\Omega})} \Omega \times \Omega$. We define the morphism $\Rightarrow : \Omega \times \Omega \rightarrow \Omega$ to be the classifying map of the equalizer of $\Omega \times \Omega \xrightarrow[\pi_1]{\vee} \Omega$.
- 5. (a) We start by defining a functor χ_{-} : Mono $(\mathcal{E}) \to \mathcal{E}/\Omega$ which sends an object $m: A \to B$ of Mono (\mathcal{E}) to its classifying map $\chi_{m}: B \to \Omega$. On morphisms, we send a pullback square



where we need to check that the triangle commutes, i.e. that $\chi_n \circ g = \chi_m$. But this follows from the uniqueness of classifying maps and the fact that the outer rectangle in the diagram



is a pullback, which is true because the left and right squares are pullbacks.

In the other direction, we define a functor $(-): \mathcal{E}/\Omega \to \text{Mono}(\mathcal{E})$. Given an object $\varphi: X \to \Omega$ of \mathcal{E}/Ω , we send it to the mono $\tilde{\varphi}: \tilde{X} \to X$ given by pulling φ back along true: $1 \to \Omega$. Given a morphism

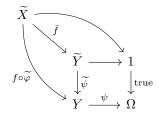
$$X \xrightarrow{f} Y$$

in the slice category \mathcal{E}/Ω , we send it to the pullback square

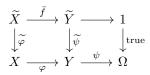
 to

$$\begin{array}{cccc}
\widetilde{X} & \xrightarrow{\bar{f}} & \widetilde{Y} \\
\widetilde{\varphi} & & & \downarrow \widetilde{\psi} \\
X & \xrightarrow{\varphi} & Y
\end{array}$$
(†)

where \bar{f} is the unique dashed map making the diagram



commute. Notice that the square (\dagger) is indeed a pullback, because the outer rectangle and right hand square in the diagram



are both pullbacks.

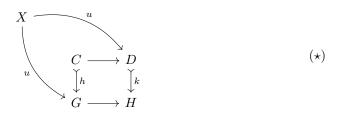
We omit the verification that these functors constitute an equivalence.

- (b) Topoi have finite colimits, so in particular \mathcal{E}/Ω has finite colimits. But Mono(\mathcal{E}) and \mathcal{E}/Ω are equivalent categories, so Mono(\mathcal{E}) has finite colimits too.
- (c) Once again we only provide a sketch of the proof. Note that the diagram

$$\begin{array}{ccc} B & \longleftarrow & A & \longrightarrow & C \\ & & & & \downarrow^g & & \downarrow^f & & \downarrow^h \\ F & \longleftarrow & E & \longrightarrow & G \end{array}$$

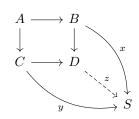
consists of two pullback squares, so we can consider its pushout in $Mono(\mathcal{E})$, which we denote by $\sigma: S \to T$. As part of the pushout property, we get pullback squares

Now given $u: X \to D$ and $v: X \to G$ making

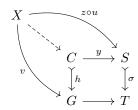


commute, we describe how to construct a map $X \to C$ which fits the diagram. From

the fact that the top square is a pushout, we get the dashed map z making



commute, where the maps x and y are as in (\ddagger). Finally, we use the right hand pullback square in (\ddagger) to get a map from X to C as the unique dashed map making the diagram



commute. We omit the verification that the dashed map makes the diagram (\star) commute and that it is the unique map to do so.